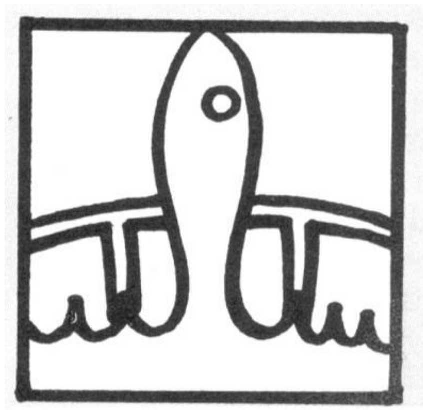
Turvy #12 Conics & Their Applications

Solution Key by David Pleacher



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S H A M U, T H E

15 1 11 12 3, 8 1 16

P E R F O R M I N G W H A L E

9 16 6 14 5 6 12 10 17 2 13 1 11 4 16

Here is the title upside-down:

M A N I N S A N D A L S

12 11 17 10 17 15 11 17 7 11 4 15

F I N D I N G A D I M E

14 10 17 7 10 17 2 11 7 10 12 16

Solutions to Problems:

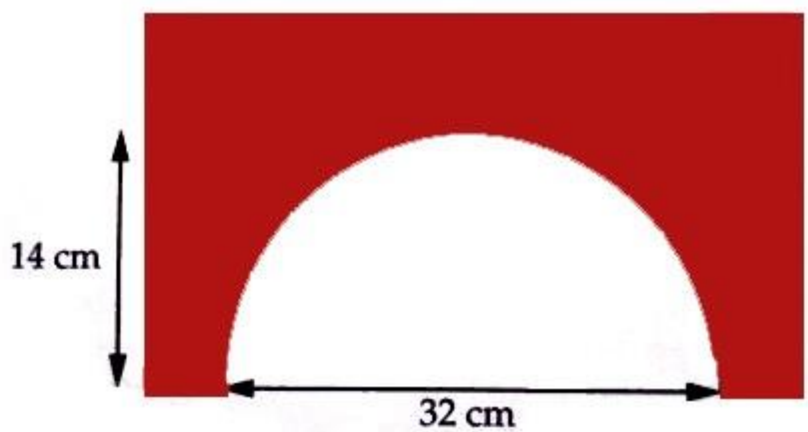
_H_ 1. Write an equation for the ellipse with vertices (4, 0) and (–2, 0) and foci (3, 0) and (–1, 0).

The center is midway between the two foci, so $(h, k) = (1, 0)$, by the Midpoint Formula. Each focus is 2 units from the center, so $c = 2$. The vertices are 3 units from the center, so $a = 3$. Also, the foci and vertices are to the left and right of each other, so this ellipse is wider than it is tall, and a^2 will go with the x part of the ellipse equation.

The equation $b^2 = a^2 - c^2$ gives us $9 - 4 = 5 = b^2$, and this is all we need to create our equation:

$$\frac{(x-1)^2}{9} + \frac{y^2}{5} = 1$$

_G_ 2. The diagram below shows the cross-section of a bridge for a model train. The highest point of the arch is 14 cm, and the span at the base of the arch is 32 cm. Find the equation of the elliptical arch of the bridge. Let the center of the ellipse be at the midpoint of the 32 cm segment, and call this the origin.



From the diagram, $a = 16$, $b = 14$.

Hence the equation of the ellipse is $\frac{x^2}{16^2} + \frac{y^2}{14^2} = 1$ or $\frac{x^2}{256} + \frac{y^2}{196} = 1$

_U_ 3. Write the equation of the parabola for which y = 1 is the directrix and F(3, –2) is the focus.

Since $|2p| = 3, \quad |p| = \frac{3}{2}.$

The parabola must turn down, so $p = -\frac{3}{2}.$

The vertex is at $\left(3, -\frac{1}{2}\right)$ and the equation is

$$(x-3)^2 = -6\left(y + \frac{1}{2}\right)$$

_L_ 4. A "whispering room" is one with an elliptically-arched ceiling. If someone stands at one focus of the ellipse and whispers something to his friend, the dispersed sound waves are reflected by the ceiling and concentrated at the other focus, allowing people across the room to clearly hear what he said. Suppose such gallery has a ceiling reaching twenty feet above the five-foot-high vertical walls at its tallest point (so the cross-section is half an ellipse topping two vertical lines at either end), and suppose the foci of the ellipse are thirty feet apart.
What is the height of the ceiling above each "whispering point"?

Since the ceiling is half of an ellipse (the top half, specifically), and since the foci will be on a line between the tops of the "straight" parts of the side walls, the foci will be five feet above the floor, which sounds about right for people talking and listening: five feet high is close to face-high on most adults.

Let's center our ellipse above the origin, so (h, k) = (0, 5). The foci are thirty feet apart, so they're 15 units to either side of the center. In particular, $c = 15$. Since the elliptical part of the room's cross-section is twenty feet high above the center, and since this "shorter" direction is the semi-minor axis, then $b = 20$. The equation $b^2 = a^2 - c^2$ gives us $400 = a^2 - 225$, so $a^2 = 625$. Then the equation for the elliptical ceiling is:

$$\frac{(x-0)^2}{625} + \frac{(y-5)^2}{400} = 1$$

We need to find the height of the ceiling above the foci. Positive numbers are easier to work with, so let's look at the focus to the right of the center. The height (from the ellipse's central line through its foci, up to the ceiling) will be the y-value of the ellipse when $x = 15$:

$$\frac{(15)^2}{625}+\frac{(y-5)^2}{400}=1$$

$$\frac{225}{625}+\frac{(y-5)^2}{400}=1$$

$$\frac{9}{25}+\frac{(y-5)^2}{400}=1$$

$144 + (y - 5)^2 = 400$
$(y - 5)^2 = 256$
$y - 5 = \pm 16$
$y = 21$

(Since we're looking for the height above, not the depth below, we ignore the negative solution to the quadratic equation).
The ceiling is **21 feet above the floor.**

_O_ 5. Determine the equation of a circle that passes through the points (3, –2), (5, 3), and (–1, 9).

There are several ways to solve this problem. One way would be to find the equations of the perpendicular bisectors of two pairs of the given points. Then solve simultaneously to get the coordinates of the center. Then use the distance formula with the center and any of the given points to determine the radius.

Another method is to use the general equation for a circle and substitute the three points into it and solve simultaneously. I thought this would be the easier way to solve for the equation.
The general form of the equation for a circle is: $x^2 + y^2 + Cx + Dy + E = 0$.
Substituting in the three points for x and y, we obtain 3 equations in 3 unknowns:

$9 + 4 + 3C - 2D + E = 0$
$25 + 9 + 5C + 3D + E = 0$
$1 + 81 - 1C + 9D + E = 0$

Then $-3C + 2D - E = 13$
$5C + 3D + E = -34$

$2C + 5D = -21$

Also, $5C + 3D + E = -34$
$C - 9D - E = 82$

$6C - 6D = 48$ or $C - D = 8$

Now solve these two equations simultaneously to get values for C and D:

6C – 6D = 48

-6C – 15D = 63

-21D = 111

$$D = -\frac{37}{7}$$

C = 8 – D

So, $C = 8 - \frac{37}{7} = \frac{19}{7}$

Now substitute these values for C and D into any of the three original equations to get

$$E = -\frac{222}{7}$$

Then, the equation for the circle is: $x^2 + y^2 + \frac{19}{7}x - \frac{37}{7}y - \frac{222}{7} = 0$

Multiplying by 7 yields $7x^2 + 7y^2 + 19x - 37y - 222 = 0$

_R_ 6. The Golden Gate Bridge is a suspension bridge in San Francisco, California. The towers are 1280 meters apart and rise 160 meters above the road. The cable just touches the sides of the road midway between the towers. What is the height of the cable 200 meters from a tower?

The cables of the bridge are parabolas. Let the vertex of one of the parabolas be (0, 0), the point at which the cable touches the side of the road. So the general form of the parabola is $x^2 = 4py$. We can solve for the equation of this parabola by substituting in the point (640, 160) – the height of one of the towers.
Hence,

$$(640)^2 = 4p(160)$$
$$409600 = 640p$$
$$p = 640 \text{ meters}$$

Then, $x^2 = 4(640)y$ or $x^2 = 2560y$ is the equation for the parabola.
To find the height of the cable 200 meters from a tower, we see that the x-coordinate in our system would be 640 – 200 = 440 m (the distance from the origin out to the cable). Now substitute in 440 m for x to find the height:

$$x^2 = 2560y$$
$$(440)^2=2560y$$
$$y = 75.625 \text{ meters}$$

__D__ 7. Determine the equation of the ellipse with center at (1, 3), major axis = 10 and parallel to the X-axis, and semi minor axis = 3.

From the given data, we have $h = 1$, $k = 3$, $a = 5$, and $b = 3$.

Thus, the equation of the ellipse is: $\dfrac{(x-1)^2}{25} + \dfrac{(y-3)^2}{9} = 1$

__T__ 8. Determine the focus of the parabola $y = 2x^2 - 3x + 5$.

Given: $y = 2x^2 - 3x + 5$

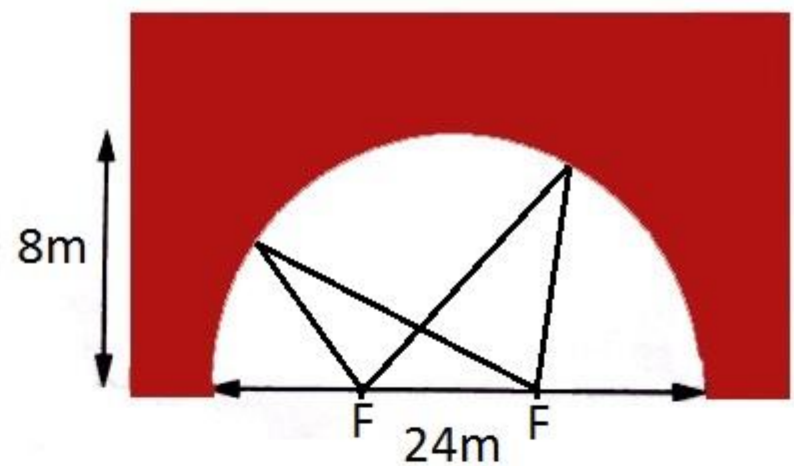
Divide by 2 and transpose: $x^2 - \dfrac{3}{2}x = \dfrac{1}{2}(y-5)$

Complete the square: $\left(x^2 - \dfrac{3}{4}\right)^2 = \dfrac{1}{2}\left(y - 5 + \dfrac{9}{8}\right)$

Simplify: $\left(x^2 - \dfrac{3}{4}\right)^2 = \dfrac{1}{2}\left(y - \dfrac{31}{8}\right)$

So the vertex is $\left(\dfrac{3}{4}, \dfrac{31}{8}\right)$ and the focus is $\left(\dfrac{3}{4}, 4\right)$

_P_ 9. A narrow arch supporting a stone bridge is in the shape of half an ellipse and is 24 meters long and 8 meters high. A person standing at one focus of the ellipse throws a rubber ball against the arch. No matter what direction the ball is thrown, it always bounces off the arch once and strikes the same point on the ground (the other focus). How far apart is the person throwing the ball from the point on the ground at which the ball strikes?



Since 2a = 24 m, then a = 12 meters.
b = 8 meters.
Use $a^2 = b^2 + c^2$ to find c.
$c^2 = 12^2 - 8^2 = 80$
c = 8.94 meters
So, 2c = 17.89 meters

_I_ 10. Determine the equation of the hyperbola with center at (1, 3), transverse axis = 8 and parallel to the X-axis, and semiconjugate axis = 3.

From the given data, we have h =1, k = 3, a = 4, and b = 3.

Therefore, the equation is: $\frac{(x-1)^2}{16} - \frac{(y-3)^2}{9} = 1$.

_A_ 11. Suppose a satellite is in an elliptical orbit, with major axis = 8840 miles and minor axis = 8832 miles, and with the center of the Earth being at one of the foci of the ellipse. Assuming the Earth has a radius of about 3960 miles, find the lowest and highest altitudes of the satellite above the Earth.

From the given information, we get a = 4420 and b = 4416. The lowest altitude will be at the vertex closer to the Earth; the highest altitude will be at the other vertex. Since we need to measure these altitudes from the focus, we need to find the value of c.

$b^2 = a^2 – c^2$
$c^2 = a^2 – b^2 = 4420^2 – 4416^2 = 35{,}344$

Then c = 188. If we set the center of my ellipse at the origin and make this a wider-than-tall ellipse, then we can put the Earth's center at the point (188, 0).

(This means, by the way, that there isn't much difference between the circumference of the Earth and the path of the satellite. The center of the elliptical orbit is actually inside the Earth, and the ellipse, having an eccentricity of *e* = 188 / 4420, or about 0.04, is pretty close to being a circle.)

The vertex closer to the end of the ellipse containing the Earth's center will be at 4420 units from the ellipse's center, or 4420 – 188 = 4232 units from the center of the Earth. Since the Earth's radius is 3960 units, then the altitude is 4232 – 3960 = 272. The other vertex is 4420 + 188 = 4608 units from the Earth's center, giving me an altitude of 4608 – 3960 = 648 units.
The minimum altitude is 272 miles above the Earth;
the maximum altitude is 648 miles above the Earth.

_M_ 12. Determine the intersection points of the circle $x^2 + y^2 = 40$ and the hyperbola xy = 12.

From xy = 12, we get $y = \frac{12}{x}$; then substituting in $x^2 + y^2 = 40$, we obtain:

$$x^2 + \frac{144}{x^2} = 40$$
$$x^4 - 40x^2 + 144 = 0$$
$$\left(x^2 - 36\right)\left(x^2 - 4\right) = 0$$
$$x = \pm 6, \pm 2$$
For $x = \pm 6, \quad y = \frac{12}{x} = \pm 2$

For $x = \pm 2, \quad y = \frac{12}{x} = \pm 6$

So, the four intersection points are: (6, 2), (–6, –2), (2, 6), and (–2, –6).

_W_ 13. Determine the equations of the circle(s) that pass through (0, –3), with radius $r=\sqrt{5}$, and with centers on the angle bisector of the first and third quadrants.

Since the angle bisector of the first and third quadrants is the line *y* = *x*, the center of the circle must have the same x- and y-coordinate. Hence *h* = *k* in the general form for a circle. Since the radius is $\sqrt{5}$, the circle(s) must have the form: $(x-h)^2+(y-h)^2=5$

Since it must pass through the point (0, –3), we can write:

$(0-h)^2+(-3-h)^2=5$

$h^2+9+6h+h^2=5$

$2h^2+6h+4=0$

$h^2+3h+2=0$

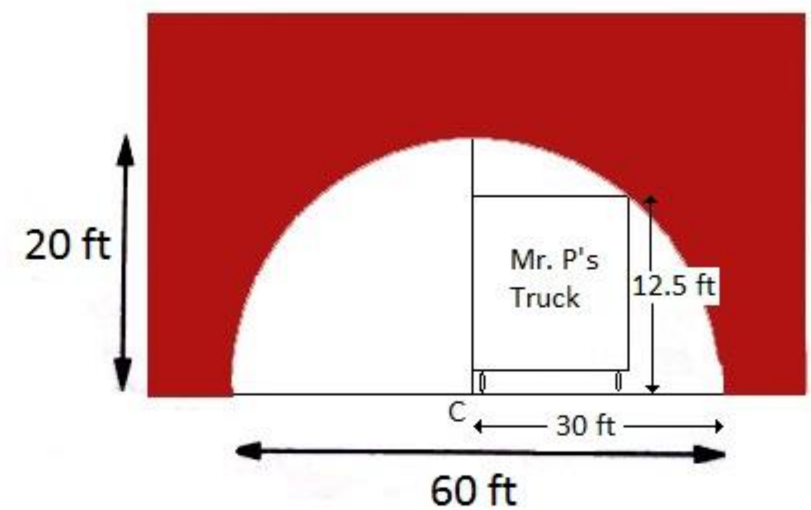
$(h+2)(h+1)=0$

So, $h=-2,-1$

Therefore, the equations of the circles are:

$(x+2)^2+(y+2)^2=5$ and $(x+1)^2+(y+1)^2=5$

_F_ 14. An arch of a bridge over a highway is semi-elliptical in shape and 60 feet across. The highest point of the arch is 20 feet above the highway. A truck that is 12 feet 6 inches tall wishes to pass through the tunnel. If one side of the truck is on the center stripe, what is the maximum width of the truck that can fit under the arch?



Let the center C be the origin (0, 0). Then a = 30 ft and b = 20 ft.

So, the equation of the ellipse is $\frac{x^2}{30^2}-\frac{y^2}{20^2}=1$ or $\frac{x^2}{900}-\frac{y^2}{400}=1$

Now substitute y = 12.5 for the height of the truck to find the maximum width x:

$$\frac{x^2}{900} - \frac{(12.5)^2}{400} = 1$$

$$\frac{x^2}{900} = 1 - \frac{156.25}{400}$$

$$\frac{x^2}{900} = .609375$$

$$x^2 = 548.4375$$

$$x = 23.42 \text{ feet}$$

_S_ 15. Determine the vertex of the parabola $4y^2 - 16x - 20y - 7 = 0$.

Given $4y^2 - 16x - 20y - 7 = 0$

Divide by 4: $y^2 - 4x - 5y - \frac{7}{4} = 0$

Complete the square: $y^2 - 5y + \frac{25}{4} = 4x + \frac{7}{4} + \frac{25}{4}$

$$\left(y - \frac{5}{2}\right)^2 = 4(x+2)$$

So, the vertex is $\left(-2, \frac{5}{2}\right)$

_E_ 16. Determine the point(s) of intersection of the circle $x^2 + y^2 = 25$ and the line x + 2y = 10.

First, solve the second equation for x: x = 10 – 2y.
Now, substitute that value for x in the first equation:

$$(10-2y)^2 + y^2 = 25$$

$$100 - 4y + 4y^2 + y^2 = 25$$

$$5y^2 - 40y + 75 = 0$$

$$y^2 - 8y + 15 = 0$$

$$(y-3)(y-5) = 0$$

$$y = 3, 5$$

So, the points of intersection are (0, 5) and (4, 3).

_N_ 17. Determine the coordinates of the foci of the ellipse

$$25x^2 + 16y^2 + 100x - 32y - 284 = 0.$$

Complete the square for x and y:

$$25\left(x^2 + 4x + 4\right) + 16\left(y^2 - 2y + 1\right) = 284 + 25(4) + 16(1)$$

$$25(x+2)^2 + 16(y-1)^2 = 400$$

Divide by 400:

$$\frac{(x+2)^2}{16} + \frac{(y-1)^2}{25} = 1$$

$a = 5, \quad b = 4, \quad \text{so } c = 3$

Therefore, the coordinates of the foci are (–2, 4) and (–2, –2).