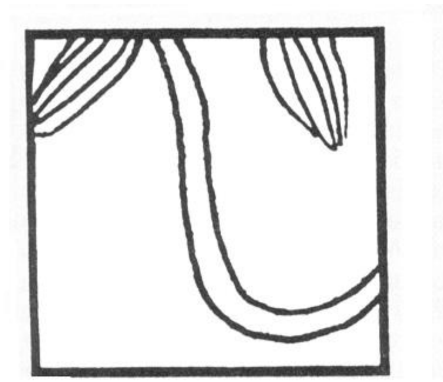
Turvy #13 Challenging Precalculus Problems

Solution Key by David Pleacher



Here is the title right-side-up:

A M U S T A C H E D M A N

20 8 9 12 19 20 1 13 15 3 8 20 7

E A T I N G S P A G H E T T I

15 20 19 17 7 16 12 18 20 16 13 15 19 19 17

Here is the title upside-down:

A G I R L W I T H P I G T A I L S

20 16 17 11 4 10 17 19 13 18 17 16 19 20 17 4 12,

S K I P P I N G R O P E

12 2 17 18 18 17 7 16 11 5 18 15

Problems:

_C_ 1. Determine the smallest value of x satisfying the equation $|x|^2 + |x| - 6 = 0$.

If x > 0, $x^2 + x - 6 = 0$ => (x – 2) (x + 3) = 0 => x = 2 (x cannot be – 3 since x>0)
If x < 0, $x^2 - x - 6 = 0$ => (x + 2) (x – 3) = 0 => x = – 2
Hence, x = – 2 is the smallest value ox that works.

_K_ 2. If $f(x) = x^3 + 3x^2 + 4x + 5$ and $g(x) = 5$

Then $g\left(f(x)\right) = g(anything) = 5$

_D_ 3. Determine the real number k for which the solution set of $|kx + 2| < 6$ is the open interval (– 1, 2).

– 6 < kx + 2 < 6
– 8 < kx < 4
So, k = – 4.

_L_ 4. If $\log_8 M + \log_8 \left(\frac{1}{6}\right) = \frac{2}{3}$, Then M =

$$\log_8 \left(\frac{M}{6}\right) = \frac{2}{3}$$

$$8^{\left(\frac{2}{3}\right)} = \frac{M}{6}$$

$$4 = \frac{M}{6}$$

$$M = 24$$

_O_ 5. If $f(x) = 2x^3 + Ax^2 + Bx - 5$ and if $f(2) = 3$ and $f(-2) = -37$

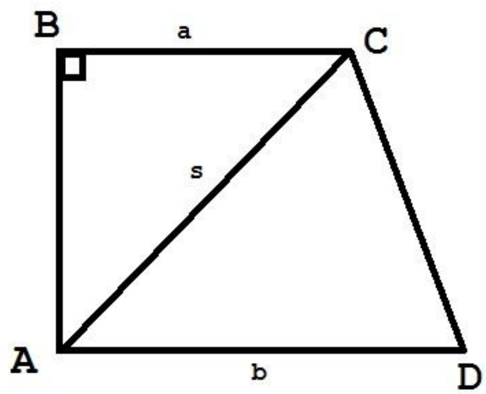
What is the value of A + B?
f(2) = 16 + 4A + 2B – 5 = 3
f(–2) = –16 + 4A – 2B – 5 = –37
Add these together to get: 8A – 10 = – 34
So A = – 3.
Then substitute back to get B = 2.
Therefore, A + B = – 1.

_Q_ 6. A ball is dropped from a height of 1 meter. It always bounces to one-half its previous height. The ball will bounce infinitely but it will travel to a finite distance.
What is the distance?
Distance = 1 + ½ + ½ + ¼ + ¼ + …
Distance = 1 + 2(1/2 + 1/4 + 1/8 + …)
Distance = 1 + 2(1) = 3

_N_ 7. In Quadrilateral ABCD,

$\overline{AB} \perp \overline{BC}, \quad \overline{AD} \| \overline{BC}, \quad m(\overline{BC}) = a, \quad m(\overline{AC}) = s, \quad m(\overline{AD}) = b,$

Determine $m(\overline{CD}) =$



By the Law of Cosines, $(CD)^2 = s^2 + b^2 - 2bs\cos(\angle A)$

$\angle BCA \cong \angle CAD$ because of alternate interior angles

$\cos(\angle \text{BCA}) = \frac{a}{s}$

$(CD)^2 = s^2 + b^2 - 2bs\left(\frac{a}{s}\right)$

$CD = \sqrt{s^2 + b^2 - 2ab}$

_M_ 8. Determine the smallest positive solution θ of the equation $2\cos^2\theta + 3\sin\theta = 0$.

$2\cos^2\theta + 3\sin\theta = 0$

$2(1 - \sin^2\theta) + 3\sin\theta = 0$

$2 - 2\sin^2\theta + 3\sin\theta = 0$

$2\sin^2\theta - 3\sin\theta - 2 = 0$

$(2\sin\theta + 1)(\sin\theta - 2) = 0$

$\sin\theta = \frac{-1}{2}$ or $\sin\theta = 2$

$\theta = 210°$

_U_ 9. Determine the **exact** value of $\sin\left(Cos^{-1}\left(-\frac{4}{5}\right)-Tan^{-1}\left(-\frac{12}{5}\right)\right)$.

Let $\theta = Cos^{-1}\left(-\frac{4}{5}\right)$

Let $\psi = Tan^{-1}\left(-\frac{12}{5}\right)$

$$\sin\left(Cos^{-1}\left(-\frac{4}{5}\right)-Tan^{-1}\left(-\frac{12}{5}\right)\right)=\sin(\theta-\psi)$$

$$\sin(\theta-\psi)=\sin\theta\cos\psi-\sin\psi\cos\theta$$

$$=\left(\frac{+3}{5}\right)\left(\frac{+5}{13}\right)-\left(\frac{-12}{13}\right)\left(\frac{-4}{5}\right)$$

$$=\frac{15}{65}-\frac{48}{65}=\frac{-33}{65}$$

_W_ 10. The sum of the first 83 nonnegative powers of *i* is _____.

Hint: $i^0+i^1+i^2+i^3+\dots+i^{82}=$

$$i^0+i^1+i^2+i^3+\dots+i^{82}=$$

$$=(1+i-1-i)+(1+i-1-i)+\dots+(1+i-1)$$

$$=(0)+(0)+\dots+(i)=i$$

_R_ 11. If $8^x=4$ and $5^{x+y}=125$, Determine y.

$$8^x=4$$

$$\left(2^3\right)^x=4$$

$$2^{3x}=2^2$$

$$3x=2$$

$$x=\frac{2}{3}$$

$$5^{x+y}=125$$

$$5^{x+y}=5^3$$

$$x+y=3$$

$$\frac{2}{3}+y=3$$

$$y=2\frac{1}{3}$$

_S_ 12. Determine the sum of the solutions of the equation $\left|x^2-16\right|=9x+6$.

$x^2 - 16 = 9x + 6$ when $x^2 - 16 > 0$
$x^2 - 9x - 22 = 0$
$(x + 2)(x - 11) = 0$
$x = -2, 11$ But -2 cannot be a solution
$-x^2 + 16 = 9x + 6$ when $x^2 - 16 < 0$
$x^2 + 9x - 10 = 0x$
$(x + 10 (x - 1) = 0$
$x = -10, 1$ But -10 cannot be a solution.
Therefore, the sum of the solutions is 11 + 1 = 12.

_H_ 13. Determine the coefficient of x^4 in the expansion $(x-2)^7 =$

Using the binomial expansion, the term with x^4 in it is given by

$${}_7C_4\, x^4(-2)^3 = 35x^4(-8) = -280x^4$$

_X_ 14. Write $\cos(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta+\theta)\\ &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta\\ &= \left(\cos^2\theta-\sin^2\theta\right)\cos\theta - (2\sin\theta\cos\theta)\sin\theta\\ &= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta\\ &= \cos^3\theta - 3\sin^2\theta\cos\theta\end{aligned}$$

_E_ 15. In a litter of 4 kittens, what is the probability that all are female?

½ x ½ x ½ x ½ = 1/16

_G_ 16. $\left(i^{17}+i^{10}\right)^3 =$

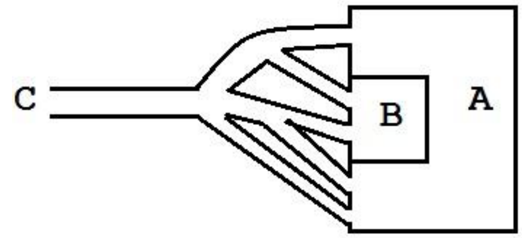
Use the template: $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$$\begin{aligned}\left(i^{17}+i^{10}\right)^3 &= \left(i^{17}\right)^3+3\left(i^{17}\right)^2 i^{10}+3i^{17}\left(i^{10}\right)^2+\left(i^{10}\right)^3\\ &= i^{51}+3i^{44}+3i^{37}+i^{30}\\ &= -i+3(1)+3i+(-1)\\ &= 2i+2\end{aligned}$$

_I_ 17. If $f(x)=\dfrac{x-1}{x}$ and $g(x)=1-x$, Then $f(g(x))=$

$$f(g(x)) = \frac{(1\text{-}x)\text{-}1}{1\text{-}x} = \frac{-x}{1-x}$$

_P_ 18. In the maze in the figure, Harry is to pick a path from C to either room A or room B. Choosing randomly at each intersection, what is the probability that Harry will enter room B?



There are five paths, three leading to A and two leading to B.
At each juncture, you have a 1/3 chance or a ½ chance of going on a certain path.
For the top path leading to A, the probability is 1/3 x ½ = 1/6.
For the next path which leads to B, the probability is 1/3 x ½ = 1/6.
For the middle path which also leads to B, the probability is 1/3 x ½ = 1/6.
For the next path which leads to A, the probability is 1/3 x ½ = 1/6.
For the bottom path leading to A, the probability is 1/3.

Therefore, the probability that Harry will enter room B is 1/6 + 1/6 = 1/3.

_T_ 19. A bouncing ball loses ¼ of its previous height each time it rebounds. If the ball is thrown up to a height of 60 feet, how many feet will it travel before coming to rest?

The key here is the word LOSES. If it loses ¼, it must gain ¾ of its previous height.
It is an infinite series and can be computed with the formula $\frac{a_1}{1-r}$.
It travels: 60 + 60 + 45 + 45 + 135/4 + 135/4 + …

The total distance = 2 (60 + 45 + 135/4 + …) = $2\left(\frac{60}{1-\frac{3}{4}}\right)=2(60\bullet 4)=480$

_A_ 20. If $\sin 2x\ \sin 3x=\cos 2x\cos 3x$, determine the smallest positive value of x that satisfies the equation.

$\sin 2x\ \sin 3x=\cos 2x\cos 3x$
$\cos 2x\cos 3x-\sin 2x\ \sin 3x=0$
$\cos(5x)=0$
$5x=90^\circ+180^\circ k$
$x=18+36k$
So $x=18^\circ$