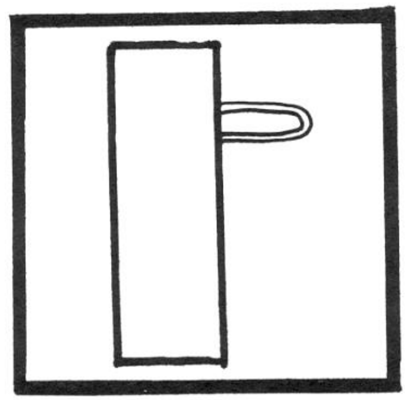
Turvy #14 Challenging Precalculus Problems

Solution Key by David Pleacher



Here is the title right-side-up:

M A N P L A Y I N G T R O M B O N E

3 17 19 6 11 17 8 10 19 7 12 2 18 3 16 18 19 20

I N P H O N E B O O T H

10 19 6 9 18 19 20 16 18 18 12 9

Here is the title upside-down:

M I D G E T P L A Y I N G T R O M B O N E

3 10 14 7 20 12 6 11 17 8 10 19 7 12 2 18 3 16 18 19 20

I N P H O N E B O O T H

10 19 6 9 18 19 20 16 18 18 12 9

If you turn the picture on its side counterclockwise, it is subject to a third interpretation:

D E C E A S E D T R O M B O N E P L A Y E R

14 20 1 20 17 4 20 14 12 2 18 3 16 18 19 20 6 11 17 8 20 2

Problems:

C 1. Determine all values of x satisfying $|x|+|x+2|=4$.

You must consider three cases:

if $x \geq 0$: $x+x+2=4 \Rightarrow 2x=2 \Rightarrow x=1$

if $-2<x<0$: $-x=x=2=4 \Rightarrow$ no solution

if $x \leq -2$: $-x-x-2=4 \Rightarrow -2x=6 \Rightarrow x=-3$

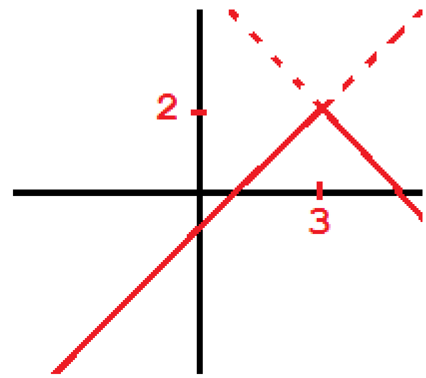
R 2. At what value of x does the function $f(x)=2-|x-3|$ have a maximum value?

You must consider two cases:

If $x-3 \geq 0$: $f(x)=2-x+3=-x+5$

If $x-3<0$: $f(x)=2+x-3=x-1$

See the graph at the right.

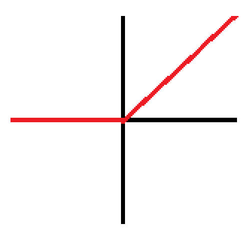


M 3. Graph the function $y=\dfrac{x+|x|}{2}$.

Again, you must consider two cases:

If $x \geq 0$, $y=\dfrac{x+x}{2} \Rightarrow y=x$

If $x<0$, $y=\dfrac{x-x}{2} \Rightarrow y=0$



S 4. Determine all solutions of $x^3 - 8 = 0$.

Factor $x^3-8=0 \Rightarrow (x-2)\left(x^2+2x+4\right)=0$

$\therefore \ x=2$

Now use the quadratic formula for $\left(x^2+2x+4\right)=0$:

$$x=\frac{-2 \pm \sqrt{4-4(1)(4)}}{2(1)}=\frac{-2 \pm \sqrt{-12}}{2(1)}=\frac{-2 \pm 2i\sqrt{3}}{2}$$

$\therefore \ x=-1 \pm i\sqrt{3}$

J 5. Determine all solutions of $x^2 - 2ix + 8 = 0$.

Move $2ix$ to the other side: $x^2 + 8 = 2ix$

Square both sides: $\left(x^2+8\right)^2 = 4i^2x^2$

$\left(x^4+16x+64\right) = 4(-1)x^2$

$x^4 + 20x + 64 = 0$

$\left(x^2+4\right)\left(x^2+16\right) = 0$

$x = \pm 2i, \pm 4i$

But since we squared both sides, we must check for extraneous roots and we find that $2i$ and $-4i$ do not work in the original equation.

P 6. Determine the real values of x satisfying the equation $(2 + 5i)x - (3 + 4i)y = -1 - 6i$.

$2x + 5ix - 3y - 4iy = -1 - 6i$

$(2x-3y) + (5x-4y)i = -1 + (-6)i$

$\therefore \quad 2x - 3y = -1$ and $5x - 4y = -6$

Solving simultaneously,

$-8x + 12y = 4$

$15x - 12y = -18$

$\therefore \quad 7x = -14$

$x = -2$

Then substituting back, $y = -1$ (not needed)

G 7. If $\sin x + \cos x = \dfrac{1}{5}$ and $0 \le x \le \pi$, Then $\tan x =$

$\cos x = \dfrac{1}{5} - \sin x \quad \Rightarrow \quad \cos^2 x = \left(\dfrac{1}{5} - \sin x\right)^2$

$1 - \sin^2 x = \dfrac{1}{25} - \dfrac{2\sin x}{5} + \sin^2 x \quad \Rightarrow \quad 2\sin^2 x - \dfrac{2\sin x}{5} - \dfrac{24}{25} = 0$

$25\sin^2 x - 5\sin x - 12 = 0 \quad \Rightarrow \quad (5\sin x + 3)(5\sin x - 4) = 0$

$\therefore \quad \sin x = \dfrac{-3}{5}$ or $\sin x = \dfrac{4}{5}$

$\sin x = \dfrac{-3}{5}$ but this yields no solutions because $0 \le x \le \pi$.

$\sin x = \dfrac{4}{5} \quad \Rightarrow \quad$ Then $\cos x = \dfrac{\pm 3}{5}$

But $\cos x$ cannot equal $\dfrac{3}{5}$ because $\sin x + \cos x = \dfrac{1}{5}$

$\therefore \quad \cos x = \frac{-3}{5}$

So, $\tan x = \frac{\sin x}{\cos x} = \frac{\frac{4}{5}}{\frac{-3}{5}} = \frac{-4}{3}$

Y 8. Determine the least positive value of Θ in degrees such that

$$\sin^2\theta + \cos^2\theta + \tan^2\theta = \frac{4}{3}.$$

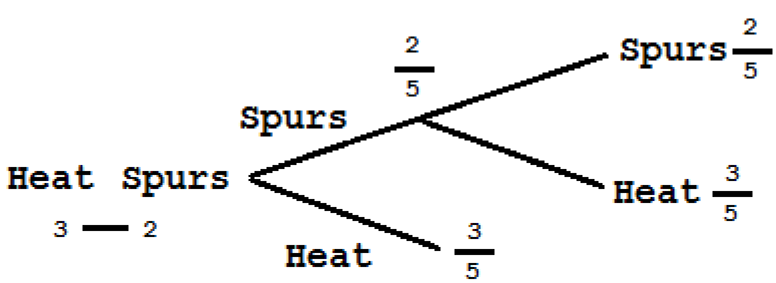
$$1 + \tan^2\theta = \frac{4}{3}$$

$$\tan^2\theta = \frac{1}{3}$$

$$\tan\theta = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$$

$$\theta = 30°$$

H 9. If the Miami Heat lead the San Antonio Spurs 3 games to 2 in a 7 game playoff, and assuming the probability of the Heat winning any game against the Spurs is 3/5, what is the probability that the Spurs will win the playoff?



So, in order for the Spurs to win the playoff series, they must win game 6 and game 7.

The probability of this happening is $\frac{2}{5} x \frac{2}{5} = \frac{4}{25}$.

The probability of the Heat winning in 6 games is $\frac{3}{5}$.

The probability of the Heat winning in 7 games is $\frac{2}{5} x \frac{3}{5} = \frac{6}{25}$.

So, the probability of the Heat winning the series is $\frac{3}{5} + \frac{6}{25} = \frac{21}{25}$.

I 10. Solve the inequality $|x-4|<5$.

Writing the inequality without absolute values gives $-5<x-4<5$, and adding 4 to every term gives the solution $-1<x<9$.

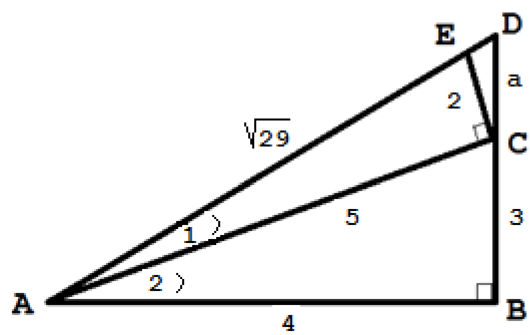
L 11. Solve the equation $2\sin^2\theta-\sin\theta=0$ for $0\le\theta<2\pi$.

Factoring $2\sin^2\theta-\sin\theta=0$ gives $\sin\theta(2\sin\theta-1)=0$

So, $\sin\theta=0$ or $\sin\theta=\frac{1}{2}$

$\therefore\theta=0,\pi,\frac{\pi}{6},\frac{5\pi}{6}$

12 – 14. In the figure at the right, $m\angle ABC=m\angle ACE=90°$, AB = 4, BC = 3, & CE = 2.



T 12. Determine AC.

Q 13. Determine AE.

D 14. Determine AD.

AC = 5 using the Pythagorean Theorem.

For the same reason, AE=$\sqrt{29}$.

We must use the Double Angle identity for tangent:

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\bullet\tan B}$$

$$\tan(\angle DAB)=\tan(\angle 1+\angle 2)$$

$$\frac{a+3}{4}=\frac{\tan(\angle 1)+\tan(\angle 2)}{1-\tan(\angle 1)\bullet\tan(\angle 2)}$$

$$\frac{a+3}{4}=\frac{\frac{2}{5}+\frac{3}{4}}{1-\left(\frac{2}{5}\bullet\frac{3}{4}\right)}$$

$$\frac{a+3}{4}=\frac{\frac{8+15}{20}}{\frac{7}{10}}$$

$$a=\frac{25}{7}$$

Use the Pythagorean Theorem in $\triangle$ABD:

$$4^2+\left(3+\frac{25}{7}\right)^2=AD^2$$

$$16+\frac{2116}{49}=AD^2$$

$$AD^2=\frac{2900}{49}$$

$$AD=\frac{10\sqrt{29}}{7}$$

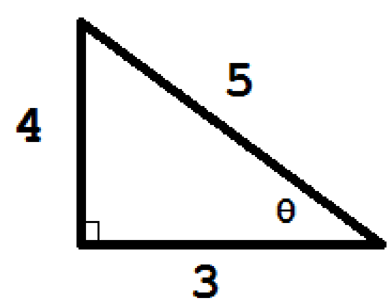
W 15. Determine $\sin\left(\text{Tan}^{-1}\frac{4}{3}\right)$.

Let $\theta=\text{Tan}^{-1}\frac{4}{3}$

So, $\tan\theta=\frac{4}{3}$

From the right triangle shown at the right,

$\sin\theta=\frac{4}{5}$, so $\sin\left(\text{Tan}^{-1}\frac{4}{3}\right)=\frac{4}{5}$



B 16. Determine $\text{Sin}^{-1}\left(\sin\frac{18\pi}{5}\right)$.

Since $\frac{18\pi}{5}$ is in the fourth quadrant,

and the range for the Sin^{-1} function is $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$,

$$\text{Sin}^{-1}\left(\sin\frac{18\pi}{5}\right)=\text{Sin}^{-1}\left(\sin\frac{-2\pi}{5}\right)=\frac{-2\pi}{5}$$

A 17. Solve the equation $e^{2x}-4e^{x}+3=0$.

Factoring $e^{2x}-4e^{x}+3=0$ gives $\left(e^{x}-3\right)\left(e^{x}-1\right)=0$

So, $e^{x}=3$ or $e^{x}=1$

Taking natural logarithms gives $x=\ln 3$ or $x=\ln 1=0$

O 18. Solve the equation $\log_3(x+5)-\log_3(x-7)=2$.

$\log_3(x+5)-\log_3(x-7)=2 \Rightarrow \log_3\left(\frac{x+5}{x-7}\right)=2$

$\frac{x+5}{x-7}=3^2 \Rightarrow x+5=9x-63 \Rightarrow 8x=68$

So, $x=\frac{68}{8}=\frac{17}{2}$

N 19. Solve the equation $x-5\sqrt{x}=-6$.

$x-5\sqrt{x}=-6 \Rightarrow x+6=5\sqrt{x}$

Squaring both sides gives $x^2+12x+36=25x$

So, $x^2-13x+36=0$

$(x-4)(x-9)=0$

$x=4,9$

E 20. If $f(x)=\frac{2x-5}{x+4}$, Determine a formula for the inverse function $f^{-1}(x)$.

$$y=\frac{2x-5}{x+4}$$

$$x=\frac{2y-5}{y+4}$$

$$xy+4x=2y-5$$

$$xy-2y=-4x-5$$

$$2y-xy=4x+5$$

$$y=\frac{4x+5}{2-x}$$

Answers (units are omitted because it would give some answers away):

A. 0, ln(3)

B. $\frac{-2\pi}{5}$

C. $-3, 1$

D. $\frac{10\sqrt{29}}{7}$

E. $\frac{4x+5}{2-x}$

F. $10\sqrt{29}$

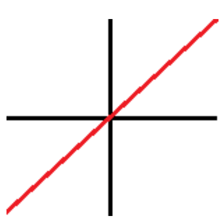
G. $\frac{-4}{3}$

H. $\frac{21}{25}$

I. $-1<x<9$

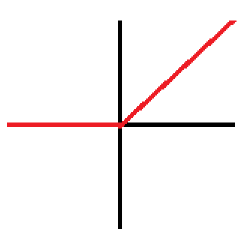
J. −2i, 4i

K.



L. $0, \frac{\pi}{2}, \pi$

M.



N. 4, 9

O. $\frac{17}{2}$

P. -2

Q. $\sqrt{29}$

R. 3

S. $2, \quad -1 \pm \sqrt{3}\,i$

T. 5

U. −3, 8

V. $\frac{-4}{5}$

W. 60

X. $\frac{4}{5}$

Y. 30

Z. None of the above