Yoplait with Areas and Volumes

Yoplait yogurt comes in two differently shaped containers. One is a truncated cone and the other is an elliptical cylinder (see photos below).

In this exercise, you will determine the volume of the Yoplait truncated cone in three different ways: (1) using geometry, (2) using the Disk Method from Calculus, and (3) using the Shell Method. You will then determine the area formula for an ellipse using calculus, and then compute the volume of the cylinder.

Each container claims to contain six ounces of yogurt, which is equivalent to 10.83 cubic inches.

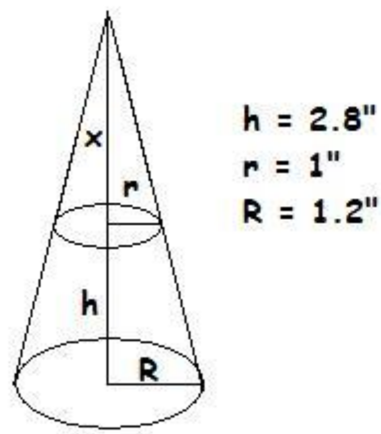
I measured the containers to the nearest tenth of an inch.

In the elliptical cylinder, the major axis was 2.5 inches, the minor axis was 2.1 inches, and it was 2.6 inches tall.

In the truncated cone, the diameter of the smaller circle (top) was 2 inches, the diameter of the larger circle (bottom) was 2.4 inches, and it was 2.8 inches tall. In the latter, the height is deceiving because the container is recessed on the bottom; the 2.8 inches is the height of the portion of the container that holds the yogurt – the actual container is over three inches tall.

(1) The Yoplait Truncated Cone Container – Using geometry to find its volume

First extend the sides of the container until it forms a cone, as pictured in the diagram below.



Let h = the height of the truncated cone.

Let x = the height of the new cone that is formed by extending the sides.
Solve for x.

Now solve for the volume of the truncated cone by subtracting the volumes of the two cones.

How does this answer compare to 10.83 cubic inches (the six ounces of yogurt)?

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(2) The Yoplait Truncated Cone Container – Using the Disk Method to find its volume

In the diagram in (1), let R represent the positive x-axis, and h + x represent the positive y-axis.

Let the intersection of those two segments represent the origin.

Now solve for the equation of the line representing the lateral edge of the cone.

Now determine the volume obtained when rotating that line about the y-axis from y = 0 to y = 2.8".

How does this answer compare to the answer obtained in (1)?

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(3) The Yoplait Truncated Cone Container – Using the Shell Method to find its volume

Now determine the volume of the truncated cone by using the Shell Method.

Note that this must be done in two steps. Find the volume from x = 0 to x = 1, and then find the volume from x = 1 to x = 1.2”. The height of the cylindrical shells from x = 0 to x = 1 is a constant, but the heights diminish from x = 1 to x = 1.2.

How does this answer compare to the answers obtained in (1) and (2)?

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(4) The Yoplait Elliptical Cylinder Container -- Find the area of an ellipse

Use calculus to determine the area of an ellipse with major axis 2a and minor axes 2b. Use the formula $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$as the formula for the ellipse, and integrate the function in the first quadrant from x = 0 to x = a, and then multiply by four.
You will need to use the method of Trigonometric Substitution and some trig identities.

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(5) The Yoplait Elliptical Cylinder Container -- Find the volume

Now, using the formula obtained in (4), multiply by the height to get the volume of the cylinder.

How does this answer compare to the answers obtained in (1), (2), and (3)?

Answer Key

1. Use similar triangles to solve for x:

$$\frac{x}{1} = \frac{2.8 + x}{1.2}$$
$$1.2x = 2.8 + x$$
$$.2x = 2.8$$
$$x = 14$$

Now determine the volumes of the two cones and subtract:

$$\text{Volume of the truncated cone} = \frac{1}{3}\pi(R)^2(H) - \frac{1}{3}\pi(r)^2(h)$$
$$= \frac{1}{3}\pi(1.2)^2(16.8) - \frac{1}{3}\pi(1)^2(14)$$
$$= 25.33 - 14.66 = 10.673 \text{ cubic inches}$$

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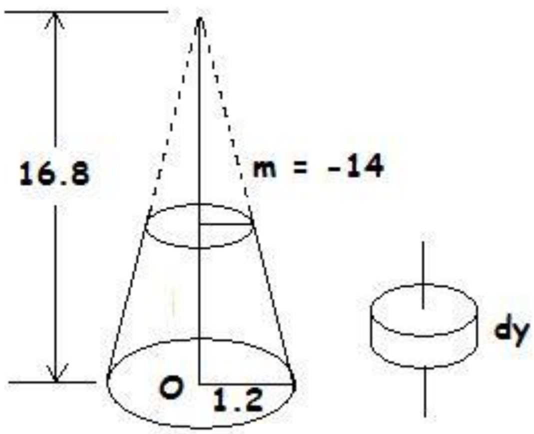
2. First, solve for the equation of the lateral edge of the cone on the right hand side.
Note that it intersects the y-axis at (0, 16.8) and that it intersects the x-axis at (1.2, 0).
Therefore, its slope is m = - 14, and the equation is y = - 14x + 16.8.

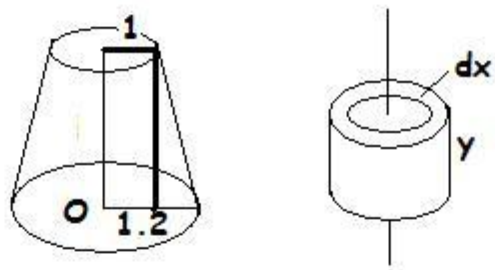


Now, we want to rotate this line about the y-axis from y = 0 to y = 2.8 (the top of the truncated cone). Note that the disks have thickness dy. Before setting up the integral for the volume, you must first solve the equation of the line for x in terms of y.

$$x = 1.2 - \frac{y}{14}$$

$$V = \int_{y=c}^{y=d} \pi\left(g(y)\right)^2 dy = \int_{y=0}^{y=2.8} \pi\left(1.2 - \frac{y}{14}\right)^2 dy = 10.673 \text{ cubic inches}$$

3. In the Shell Method, rotation is still about the y-axis but the function is in terms of x, and thickness is therefore dx. Also, note that this must be done in two steps because the heights of the shells are not found in the same way. From x = 0 to x = 1, the height of the shells is a constant 2.8”; but from x = 1 to x = 1.2, the heights of the shells are from the x-axis to the line y = -14x + 16.8.



$$V = \int_{x=a}^{x=b} 2\pi x Y\, dx + \int_{x=b}^{x=c} 2\pi x y\, dx$$

$$V = \int_{x=0}^{x=1} 2\pi x(2.8)\, dx + \int_{x=1}^{x=1.2} 2\pi x(-14x+16.8)\, dx$$

$$V = 8.796459 + 1.87657 = 10.673 \text{ cubic inches}$$

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(4) To determine the area of an ellipse with major axis 2a and minor axes 2b, use the formula $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$as the formula for the ellipse, and integrate the function in the first quadrant from x = 0 to x = a, and then multiply by four.

You must first solve the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$for y in terms of x in the first quadrant:

$$y=+\sqrt{b^2-\frac{b^2}{a^2}x^2}$$

Now, find the area of the ellipse by integrating this function from x = 0 to x = a:

$$A=\int_{x=0}^{x=a} f(x)\,dx = \int_{x=0}^{x=a} \sqrt{b^2-\frac{b^2}{a^2}x^2}\,dx$$

$$A=\int_{x=0}^{x=a} b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

Now use the Method of Trig Substitution to integrate:

Let $x = a\sin\theta$

Then $dx = a\cos\theta\, d\theta$

$$A = \int_{x=0}^{x=a} b\left(\sqrt{1-\frac{(a\sin\theta)^2}{a^2}}\right) a\cos\theta\, d\theta$$

$$A = \int_{x=0}^{x=a} b\left(\sqrt{1-\sin^2\theta}\right) a\cos\theta\, d\theta$$

$$A = \int_{x=0}^{x=a} ab\cos^2\theta\, d\theta$$

To integrate, use the trig identity called the double angle formula:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$A = \int_{x=0}^{x=a} ab\left(\frac{1+\cos 2\theta}{2}\right) d\theta = ab\int_{x=0}^{x=a}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) d\theta$$

$$A = ab\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^a = ab\left(\frac{1}{2}\left(Sin^{-1}\left(\frac{x}{a}\right)\right) + \frac{1}{4}\left(2\sin\theta\cos\theta\right)\right)\Bigg|_0^a$$

$$A = ab\left(\frac{1}{2}\left(Sin^{-1}\left(\frac{x}{a}\right)\right) + \frac{1}{2}\left(\frac{x}{a}\right)\left(\frac{\sqrt{a^2-x^2}}{a}\right)\right)\Bigg|_0^a$$

$$A = ab\left(\frac{1}{2}\bullet\frac{\pi}{2}+0\right) - ab(0+0) = \frac{1}{4}\pi ab$$

Now multiply by four, to obtain the formula for the area of an ellipse:

$$A = \pi ab$$

(5) Now to determine the volume of the elliptical cylinder, multiply the area of the base by the height of the cylinder.

$$V = \pi abh$$
$$V = \pi(1.25)(1.05)(2.6)$$
$$V = 10.72 \text{ cubic inches}$$

Six ounces of yogurt is equivalent to 10.83 cubic inches.

In computations #1, 2, and 3, we obtained 10.673 cubic inches.

In computation #4 and 5, we obtained 10.72 cubic inches.

The error probably lies in the measurements taken, but all computations were close.