The Mathematics of the Soda Pop Can

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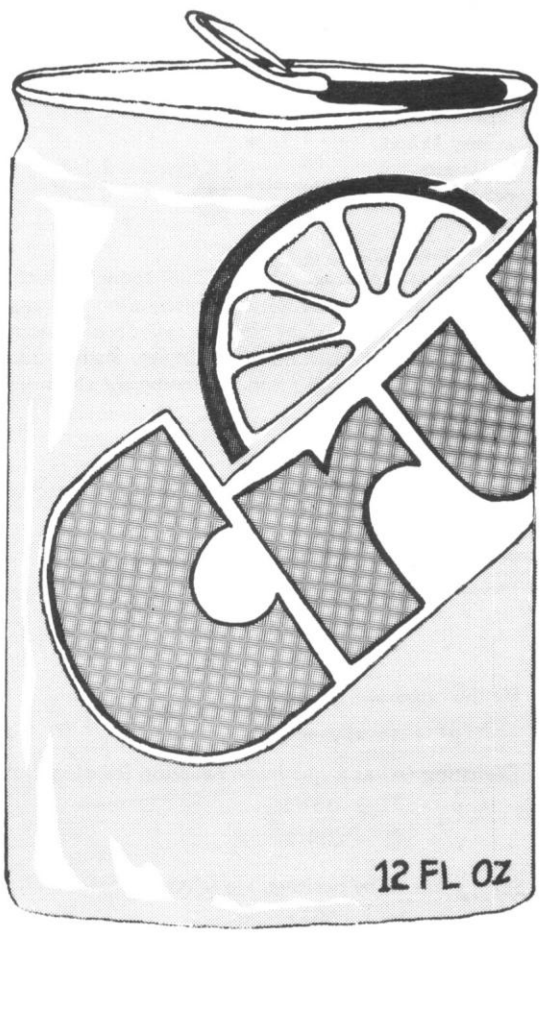


This HiMAP Pull-Out Section explores the mathematics behind how a company chooses the size and shape of its product container. Is the can the most efficient shape for the product? Is the container the cheapest way the company can make its product? Turn the page and find out.

The next time you are at the supermarket, look at the various sizes and shapes of food containers. Although the sizes vary from small one-ounce containers of imported cheese to twenty-five pound boxes of soap, the number of shapes are essentially only two; rectangular parallelpipeds and right circular cylinders. In the examples and exercises (YOU TRY IT) that follow, you will see how mathematics can be used to solve some interesting problems related to these two shapes, with a special emphasis on the 12-ounce soda pop can.

Let's first consider some containers that are right circular cylinders. The following table presents the approximate dimensions and net weights of some popular items found in many homes:

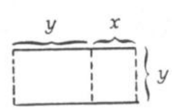
Table 1

Name of Item	Net Weight	Height	Diameter
Campbell's Soup	10.75 oz.	4″	2 5/8″
Libby's Pumpkin	29 oz.	4 5/8″	4″
Bumble Bee Tuna	6.5 oz.	1 3/16″	3 3/8″
LaChoy Water Chestnuts	8 oz.	2 1/8″	3 3/8″
Soda Pop	12 oz.	4 7/8″	2 1/2″

If one takes a photograph of each of these products, the resulting picture is a two-dimensional representation (rectangle) of a three-dimensional object (right circular cylinder). Some rectangles are "nicer to look at" than other rectangles. Such rectangles are said to possess the "golden mean," whereby the ratio of length/width has the property that

$$(y + x)/y = y/x, \qquad \textbf{(A)}$$

where x and y are the dimensions of the two rectangles in the following figure:



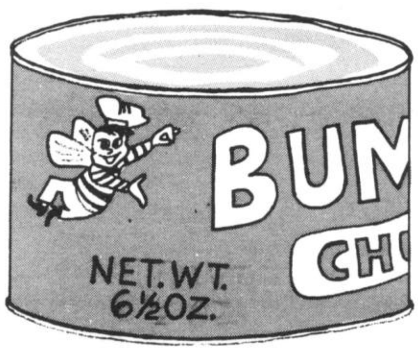
By the quadratic formula,

$$y^2 + (-x)y + (-x^2) = 0.$$

Rewriting (**A**) as a quadratic equation (in y), we obtain

$$y = x(1 + \sqrt{5})/2$$
$$y = x(1 - \sqrt{5})/2.$$

Since y has to be positive, we select the first solution,



As mentioned above, some rectangles are "nicer to look at" than others and such rectangles possess the "golden mean," whereby

length/width = 1.618034.

Some examples of such rectangles are the following: index cards (3" by 5" and 5" by 8"), credit cards, some of the art by Leonardo da Vinci, and a front view or side view of the Parthenon in Athens, Greece.

Let's now see if any of the products listed in **Table 1** also have the "golden mean."

You Try It #1. Consider the items listed in **Table 1** above and determine which of these possess the "golden mean" when looking at the ratio of height/diameter.

Let's now concentrate on the 12-ounce soda pop can listed in Table 1. With a height of approximately 4 7/8 inches and a diameter of approximately 2 1/2 inches, it does NOT possess the "golden mean" as

height/diameter = (4 7/8)/(2 1/2) = 1.95.

Does this mean that the can has no esthetic value; that it does not have the "nice looks" that other cans have that possess the "golden mean"? After all, one of the major ways of selling a product is to exploit its looks.

Or is there an economic reason for making the typical soda pop can with dimensions of 4 7/8 inches in height and 2 1/2 inches in diameter? With these dimensions are we guaranteed to get a can that is made from the least amount of aluminum? As we shall soon see, this is NOT the case; that is, there are other cans that use less aluminum and still hold 12 fluid ounces. To see why this is so, we first need to look at "the arithmetic and geometric means" inequality.

Theorem: *The Arithmetic/Geometric Means Inequality.*

Suppose that x and y are two non-negative numbers and we define the arithmetic mean of x and y to be $= (x + y)/2$ and the geometric mean of x and y to be $= \sqrt{xy}$. Then

$$(x + y)/2 \geq \sqrt{xy}, \qquad \text{(B)}$$

where equality is true only if $x = y$.

Proof: For non-negative numbers x and y, both $\sqrt{x}$ and $\sqrt{y}$ are real numbers, and the square of their difference is a non-negative real number; that is,

$$(\sqrt{x} - \sqrt{y})^2 \geq 0, \qquad \text{(C)}$$

$$x - 2\sqrt{xy} + y \geq 0,$$

or,
$$(x + y)/2 \geq \sqrt{xy}.$$

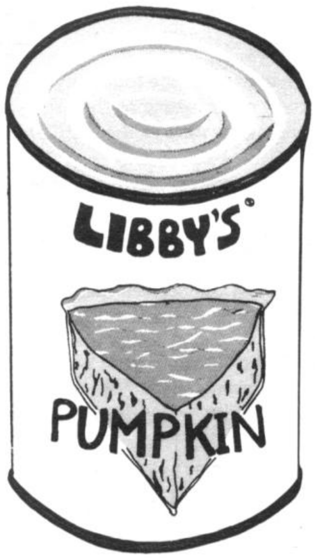
The left side of (C) = 0 if $x = y$, and if

$$(x + y)/2 = \sqrt{xy},$$

then $x = y$. Otherwise,

$$(x + y)/2 > \sqrt{xy}, \text{ if } x \neq y.$$

Let's look at some problems that can be solved with this important relationship.



Example #1: Find two positive numbers whose product is 100 and whose sum is as small as possible.
Solution: Let x and y be the two positive numbers that we seek so that $xy = 100$ and we want to find the smallest value of their sum $(x + y)$. Using (B) we obtain $(x + y)/2 \geq \sqrt{xy}$ which $= \sqrt{100} = 10$, or, $(x + y) \geq 20$. Since the smallest value of the sum is 20, then $x = y = 10$.

You Try It #2: Suppose that you want to enclose a plot of land with a rectangular fence so that the enclosed area is 100 square miles and you use the least amount of fencing. Determine the dimensions of the plot of land.

You Try It #3: The Arithmetic Mean/Geometric Mean Inequality is a relationship between two non-negative numbers. Make a conjecture for a similar relationship for three non-negative numbers.

If your conjecture in YOU TRY IT #3 above is

$$(x + y + z)/3 \geq \sqrt[3]{xyz}, \qquad \text{(D)}$$

then you are correct! (Try some non-negative numbers for x, y, and z and show that this statement is correct for those values.)

Let's see how to develop (D) by using the fact that when x and y are non-negative numbers,

$$(x + y)/2 \geq \sqrt{xy}. \qquad \text{(E)}$$

Since $\sqrt{xy} > = 0$, then we can square both sides of (E) and simplify to obtain

$$(x + y)^2 \geq 4xy,$$

or

$$(x^2 + y^2) \geq 2xy. \qquad \text{(F)}$$

Suppose that a, b, and c are three non-negative numbers, so that from (F), we obtain

$$a^2 + b^2 \geq 2ab,$$

$$a^2 + c^2 \geq 2ac,$$

$$b^2 + c^2 \geq 2bc.$$

Then $(a^2 + b^2 + c^2) \geq (ab + ac + bc)$,
or, $(a^2 + b^2 + c^2 - ab - ac - bc) \geq 0. \qquad \text{(G)}$

Since a, b, and c are non-negative numbers, then $(a + b + c) \geq 0$, so that (G) can be written as

$$(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) \geq 0. \qquad \text{(H)}$$

Removing the parentheses from (H) and simplifying (do it; don't take my word for it!), we obtain

$$a^3 + b^3 + c^3 - 3abc \geq 0,$$

or, $(a^3 + b^3 + c^3)/3 \geq abc. \qquad \text{(I)}$

Letting $a = \sqrt[3]{x}$, $b = \sqrt[3]{y}$, and $c = \sqrt[3]{x}$, so that x, y, and z are three non-negative numbers, then (I) becomes

$$(x + y + z)/3 \geq \sqrt[3]{xyz}.$$

Notice that the equality is true only when $x = y = z$. Let's now use (D) to solve some problems.

Example #2: Find positive real numbers a and b, so that the following expression has a minimum value:

$$24/a + 6/b + 12ab.$$

Solution: To solve this problem using (D), we obtain $(24/a + 6/b + 12ab)/3 \geq \sqrt[3]{(24/a)(6/b)(12ab)} = \sqrt[3]{12^3} = 12,$

or, $(24/a + 6/b + 12ab) \geq 36.$

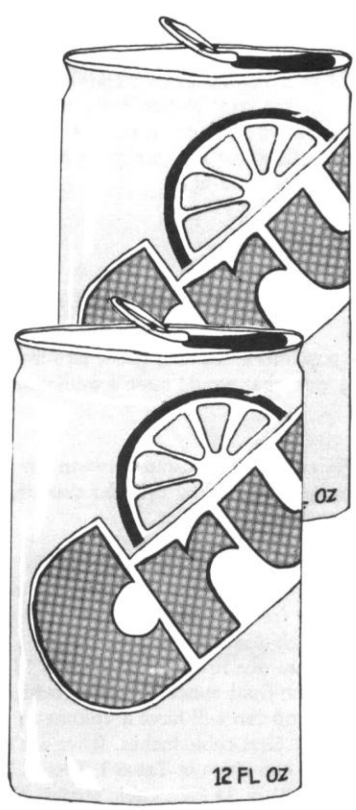
Hence, the minimum value is 36 which occurs when

$$24/a = 6/b = 12ab.$$

From the first two fractions we see that $a = 4b$, while from the second and the third fractions, $2ab^2 = 1$. Forming a single equation we obtain

$$2(4b)(b^2) = 1, \text{ or } b = 1/2,$$

which means that $a = 2$.
Thus, if $a = 2$ and $b = 1/2$, then 36 is the minimum value.



We can now show that the typical 4 7/8 inches (height) by 2 1/2 inches (diameter) soda pop can is NO the most economical can in terms of the amount of aluminum that is used. That is, there are other dimensions for such a can that will hold the same amount of soda (that is, have a fixed or constant volume) but use less aluminum. To keep our calculations to a bare minimum, let's assume that a typical 12-ounce soda pop can is a right circular cylinder with radius of r units and a height of h units, and that the top, the bottom, and the side are all made of aluminum. (If you look at a typical soda pop can, the top and bottom have a different gauge of aluminum from that used on the side, and that the bottom of the can is often not a flat circular disk.) Using the formulas for the volume of a right circular cylinder V and the total surface area S, we have

$$V = \pi r^2 h \text{ and } S = 2\pi r^2 + 2\pi rh.$$

Since the first equation can be written as $h = V/(\pi r^2)$, then the second equation becomes

$$S = 2\pi r^2 + 2\pi r(V/(\pi r^2)),$$
or,
$$S = 2\pi r^2 + 2V/r.$$

To obtain a minimum value for S (for a fixed or constant volume V), we would like to obtain a minimum value for

$$2\pi r^2 + 2V/r. \qquad \text{(J)}$$

If we use (B) on (J), we see that

$$(2\pi r^2 + 2V/r)/2 \geq \sqrt{(2\pi r^2)(2V/r)},$$
or,
$$S = (2\pi r^2 + 2V/r) \geq 4\sqrt{(\pi rV)},$$

where $4\sqrt{(\pi rV)}$ is NOT a constant since r is NOT a constant.

Since we were not able to use (B) on (J) to minimize the surface area S, can we use (D) on (J)? In its present form, we cannot use (D) on (J) since (J) has but only two terms. However, if we rewrite (J) as

$$2\pi r^2 + V/r + V/r, \tag{K}$$

then (J), or now (K), has three terms so now let's apply (D) on (K). Then

$$(2\pi r^2 + V/r + V/r)/3 \geq \sqrt[3]{(2\pi r^2)(V/r)(V/r)},$$
$$(2\pi r^2 + V/r + V/r)/3 \geq \sqrt[3]{(2\pi V^2)},$$
$$S = (2\pi r^2 + V/r + V/r) \geq 3\sqrt[3]{(2\pi V^2)},$$

where $3\sqrt[3]{(2\pi V^2)}$ is a constant. Hence,

$$S \geq 3\sqrt[3]{(2\pi V^2)}. \tag{L}$$

This means that the surface area S will have a minimum value of $3\sqrt{(2\pi V^2)}$ when the three terms of (K) are equal; that is

$$2\pi r^2 = V/r = V/r,$$

so that $V = 2\pi r^3$. Since $V = \pi r^2 h$, then

$$2\pi r^3 = \pi r^2 h,$$
or,
$$h = 2r.$$

Hence, the surface area of a right circular cylinder is a minimum value when the height, h, is the same as the diameter, $2r$.

The 12-ounce soda pop can (whose height $h = 4\ 7/8$ inches and whose diameter is $2r = 2\ 1/2$ inches) is certainly NOT a can that has a minimum surface area. Notice that none of the other cans listed in Table 1 have a minimum surface area. Also, note that any can whose surface area is a minimum does NOT possess the "golden mean" since ratio of

$$\text{height/diameter} = h/2r = h/h = 1.$$

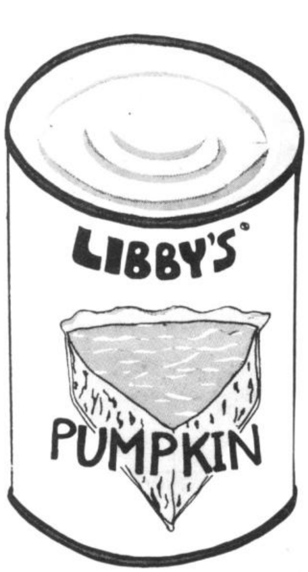
You Try It #4: (Optional: for calculus students only.) Verify the above result, $h = 2r$, using either the First Derivative Test or the Second Derivative Test.

If we pour the soda pop into a drinking glass, let's investigate the nature of a drinking glass that would have a minimum surface area.

You Try It #5: Determine the relationship between the height h and the base radius r of a right circular cylinder drinking glass so that the total surface area is a minimum.

Were you surprised with the results of **You Try It #5**? For a minimum surface area, did you expect the dimensions of the drinking glass to be different from (or the same as) the dimensions of the soda pop can with a minimum surface area?

On a typical soda pop can, one finds the statement "12 fluid ounces (or) 354 ml." Since one fluid ounce = 1.8047 cubic inches, then a typical soda pop can will have a volume of 12(1.8047 cubic inches) = 21.6564 cubic inches. If we use the dimensions for the soda pop can given in Table 1, then $V = \pi r^2 h = \pi(1\ 1/4)^2(4\ 7/8) = 23.9301$ cubic inches. The two volumes, 21.6564 and 23.9301, differ because a typical soda pop can is often not a right circular cylinder; we have assumed that it is with all of these calculations though.



Using the volume of a 12-fluid ounce soda pop can to be 21.6564 cubic inches, we will now determine the dimensions of the can and its minimum surface area. Since $h = 2r$, then

$$V = \pi r^2 h = 2\pi r^3,$$
$$21.6564 = 2\pi r^3, \text{ so that}$$
$$r = \sqrt[3]{(21.6564/2\pi)},$$
or,
$$r = 1.51055 \text{ inches and } h = 3.0211 \text{ inches.}$$

Then the surface area of the can is

$$S = 2\pi rh + 2\pi r^2,$$
$$S = 2\pi(1.51055)(3.0211) + 2\pi(1.51055)^2,$$
$$S = 43.010187 \text{ square inches.} \qquad \text{(M)}$$

Hence the minimum surface area is 43.01087 square inches, while the surface area for the can listed in Table 1 is

$$S = 2\pi(1\ 1/4)(4\ 7/8) + 2\pi(1\ 1/4)^2,$$
or,
$$S = 48.105637 \text{ square inches.}$$

Even though these two surface areas are only approximate values, notice that they differ by more than five square inches.

You Try It #6: Use the results of **(L)** to obtain **(M)**.

You Try It #7: Suppose that you sell a 12-ounce soft drink product in a plastic container whose shape is a rectangular parallelpiped. **(a)** Determine the dimensions of such a container so that its surface area is minimum. **(b)** Determine what the surface area is. **(c)** Compare the surface area for this container with that of the 12-ounce soda pop can of minimum surface area.

Although the cube-shaped soda pop container has less surface area than the present typical soda pop can, its surface area is more than the can with minimum surface area. This should make one wonder why the typical 12-ounce soda pop can is presently used. In terms of looks, it does NOT possess the "golden mean"; in terms of economics, it is NOT made from the least amount of aluminum. Maybe we need to look at the physiology of the human head. Is this can "easier to hold" than other cans?

Some Answers to "You Try It"

1. Name of Item	Height	Diameter	Height/ Diameter
Campbell's Soup	4"	2 5/8"	1.52
Libby's Pumpkin	4 5/8"	4"	1.16
Bumble Bee Tuna	1 3/16"	3 3/8"	0.35
LaChoy Water Chestnuts	2 1/8"	3 3/8"	0.63
Soda Pop	4 7/8"	2 1/2"	1.95

Of the above items, only the Campbell's soup can comes "close" to possessing the "golden mean."

2. If x and y represent the length and width respectively, then the perimeter $= 2x + 2y$ and $xy = 100$ sq. ft. Since

$$(x + y)/2 \geq \sqrt{xy},$$

then $(2x + 2y) \geq 4\sqrt{xy} = 4\sqrt{100} = 40$, or, $(x + y) \geq 20$. Since the smallest value of $(x + y)$ will occur if $x = y$, then $x = y = 10$ miles.

3. Make a conjecture; don't look here for an answer!

5. For such a right circular cylinder, we have

$$S = \pi r^2 + 2\pi rh \text{ and } V = \pi r^2 h, \text{ or } h = V/(\pi r^2).$$

Then,

$$S = \pi r^2 + 2\pi r(V/\pi r^2),$$
$$S = \pi r^2 + 2V/r,$$
$$S = \pi r^2 + V/r + V/r,$$

$$S/3 = (\pi r^2 + V/r + V/r)/3 \geq \sqrt[3]{(\pi r^2)(V/r)(V/r)},$$

so that,

$$S \geq 3\sqrt{(\pi V^2)}, \text{ where } 3\sqrt{(\pi V^2)} \text{ is a constant.}$$

Hence, S is a minimum when

$$\pi r^2 = V/r = V/r, \text{ so that } V = \pi r^3.$$

Since $V = \pi r^2 h$, then $\pi r^2 h = \pi r^3$, or $r = h$.
Thus, the radius $r =$ the height h in order that the surface area be a minimum.

6. If $V = 21.6564$ cubic inches, then by using **(L)**, we have

$$S = 3\sqrt[3]{(2\pi)(21.6564)^2} = 43.0102047 \text{ square inches.}$$

7. (a) Let x, y, and z represent the length, width, and height respectively of a rectangular parallelpiped, so that

$S = 2xy + 2yz + 2xz$ and $V = xyz$, which is a constant. Then,

$$S/3 = (2xy + 2yz + 2xz)/3 \geq \sqrt[3]{(2xy)(2yz)(2xz)},$$

so that,

$$S \geq (3)(2)\ \sqrt[3]{x^2y^2z^2} = 6\ \sqrt[3]{V^2}, \text{ which is a constant.}$$

In order that S be a minimum, then $2xy = 2yz = 2xz$, or,

$$x = y = z.$$

Hence, the rectangular parallelpiped must be a cube in order that it has a minimum surface area. Since a 12-fluid ounce container has a volume of 21.6564 cubic inches, then

$$V = xyz = x^3 = 21.6564,$$
or,
$$x = y = z = 2.7873751 \text{ inches.}$$

(b) Then $S = 6(2.7873751)^2 = 46.61676$ square inches, or $S = 6\ \sqrt[3]{(21.6564)^2} = 46.616751$ square inches.

(c) For the 12-ounce soda pop can of minimum surface area, we saw that its surface area $= 43.010187$ square inches which is more than 3 1/2 square inches less than the above amount.