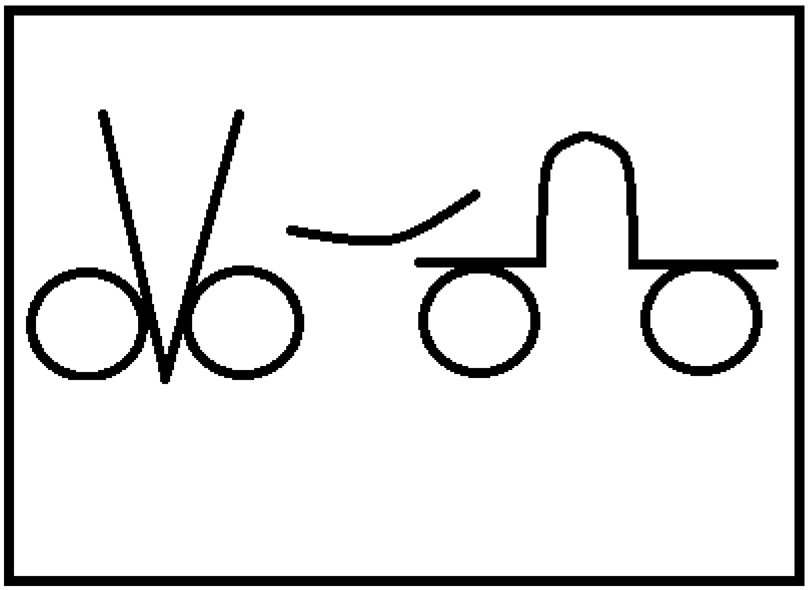
Droodle for Derivatives

Answer Key by David Pleacher

Can you name this droodle?



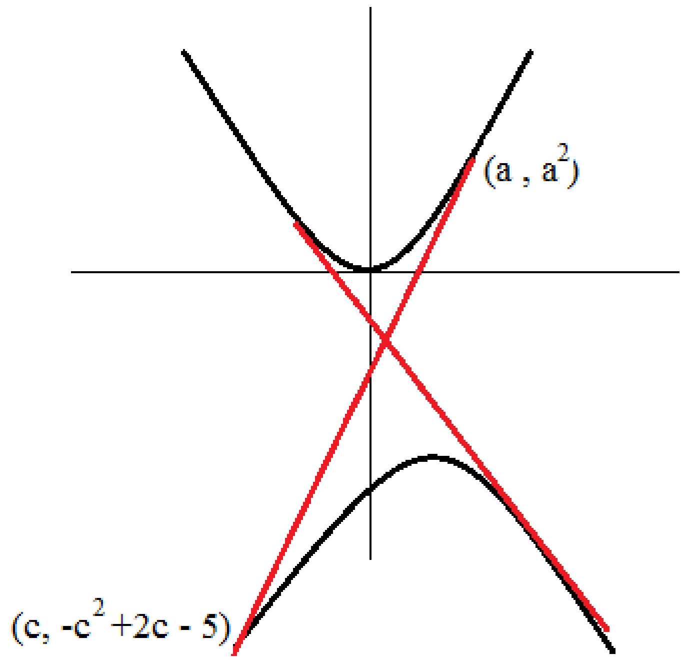
A		V	O	L	T	W	A	G	O	N		P	U	L	L	I	N	G		A
10		3	15	2	9	4	10	8	15	7		14	11	2	2	12	7	8		10

M	O	B	I	L	E		O	H	M
1	15	5	12	2	6		15	13	1.

1. M
2. L
3. V
4. W
5. B
6. E
7. N
8. G
9. T
10. A
11. U
12. I
13. H
14. P
15. O

1. Graph the two parabolas $y = x^2$ and $y = -x^2 + 2x - 5$. find the equations of the lines that are simultaneously tangent to both parabolas.

M. $y = -2x - 1$ and $y = 4x - 4$



Let a be the x-coordinate of the point of tangency on $y = x^2$ and let c be the x-coordinate of the point of tangency on $y = -x^2 + 2x - 5$. See diagram above.

The slope at any point on $y = x^2$ is given by $\frac{dy}{dx} = 2x$ and the slope at any point on $y = -x^2 + 2x - 5$ is given by $\frac{dy}{dx} = -2x + 2$.

So, the slope of a tangent line to both curves can be expressed in three different ways:

(1) slope = 2a (on $y = x^2$)

(2) slope = −2c + 2 (on $y = -x^2 + 2x - 5$

(3) slope = $\frac{a^2 + c^2 - 2c + 5}{a - c}$ using the slope formula for (a, a^2) and (c, $-c^2 + 2c - 5$)

Since these expressions must be equal, we may set any two of them equal.

Setting #1 and #2 equal, we obtain $2a = -2c + 2$ or $a + c = 1$ or $a = 1 - c$

Setting #1 and #3 equal, we obtain $2a = \frac{a^2 + c^2 - 2c + 5}{a - c}$

Then $2a(a - c) = a^2 + c^2 - 2c + 5$

So, $2a^2 - 2ac = a^2 + c^2 - 2c + 5$

Simplifying, we obtain $a^2 - 2ac = c^2 - 2c + 5$

Now substitute $a = 1 - c$ to obtain: $(1 - c)^2 - 2(1 - c)c = c^2 - 2c + 5$

Simplifying, we obtain: $2c^2 - 2c - 4 = 0$

$c^2 - c - 2 = 0$

$(c + 1)(c - 2) = 0$

$c = -1$ and 2

We get two answers for c because there are two tangents to the curve $y = -x^2 + 2x - 5$ which are also tangent to the curve $y = x^2$.

CASE 1: When $c = -1$, $a = 2$.

This gives us two points $(-1, -8)$ and $(2, 4)$ and the slope $m = 4$.

The equation of this tangent is $y - 4 = 4(x - 2)$ or $y = 4x - 4$.

CASE 2: When $c = 2$, $a = -1$.

This gives us two points $(2, -5)$ and $(-1, 1)$ and the slope $m = -2$.

The equation of this tangent is $y - 1 = -2(x + 1)$ or $y = -2x - 1$.

2. Determine the value of k so that the line $y = 5x - 4$ is tangent to the graph of the function $f(x) = x^2 - kx$. L. $-1, -9$

Since the line and the parabola are tangent, then $x^2 - kx = 5x - 4$ (share a point); and the slopes must be the same, too: $2x - k = 5$.

Therefore, $k = 2x - 5$.

Substitute this value of k into the first equation:

$x^2 - (2x - 5)x = 5x - 4$

$x^2 - 2x^2 + 5x = 5x - 4$

$-x^2 = -4$

so $x = 2$ or -2

When $x = 2$, $k = -1$

When $x = -2$, $k = -9$.

3. An object travels along a line so that its distance traveled in inches after t seconds is $s(t)=\sqrt{2t-1}$. Determine the instantaneous velocity after 5 seconds.

V. $\frac{1}{\sqrt{11}}$ in/sec

The velocity = $\frac{ds}{dt}=\frac{1}{2}(2t+1)^{\frac{-1}{2}}\bullet 2=\frac{1}{\sqrt{2t+1}}$

When t = 5 sec, v = $\frac{1}{\sqrt{11}}$ in/sec

4. Given $y=\sin^2(3x)$, Determine $\frac{d^2y}{dx^2}$.

W. $18\cos 6x$

$$
\begin{aligned}
y &= \sin^2(3x) \\
\frac{dy}{dx} &= 2(\sin 3x)\bullet\frac{d}{dx}(\sin 3x) \\
&= 2(\sin 3x)\bullet(\cos 3x)\bullet 3 \\
&= 6(\sin 3x)\bullet(\cos 3x) = 3\sin(6x) \quad \text{(double angle identity)}
\end{aligned}
$$

$$
\frac{d^2y}{dx^2} = 3\cos(6x)\bullet 6 = 18\cos 6x
$$

Of course, you could do it the hard way and take the derivative of $6(\sin 3x)\bullet(\cos 3x)$ using the product rule.

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(6(\sin 3x)\bullet(\cos 3x)\right) \\
&= 6\bullet\frac{d}{dx}\left((\sin 3x)\bullet(\cos 3x)\right) \\
&= 6\bullet\left((\sin 3x)\bullet\frac{d}{dx}(\cos 3x)+(\cos 3x)\bullet\frac{d}{dx}(\sin 3x)\right) \\
&= 6\bullet\left((\sin 3x)\bullet(-\sin 3x)\bullet 3+(\cos 3x)\bullet(\cos 3x)\bullet 3\right) \\
&= 6\bullet\left(3\cos^2 3x-3\sin^2 3x\right) \\
&= 18\bullet\left(\cos^2 3x-\sin^2 3x\right) = 18\cos 6x
\end{aligned}
$$

5. Determine $\frac{dy}{dx}$ if $x\sin y = 1$. B. – tan y / x

Use the product rule and implicit differentiation:

$$x[\cos y \bullet y'] + \sin y(1) = 0$$

$$x[\cos y \bullet y'] = -\sin y$$

$$y' = \frac{-\sin y}{x\cos y} = -\frac{\tan y}{x}$$

6. Find $\frac{dy}{dx}$ for the parametric equations x = 3t + 1 and y = 2t – 1. E. $\frac{2}{3}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dt} = 2 \quad \text{and} \quad \frac{dx}{dt} = 3$$

$$\therefore \frac{dy}{dx} = \frac{2}{3}$$

7. Find the equation of the normal line to f(x) = e^{2x} at (0,1). N. $y = \frac{-1}{2}x + 1$

$f'(x) = 2e^{2x}$ and $f'(0) = 2e^{2(0)} = 2$, which is the slope of the tangent.

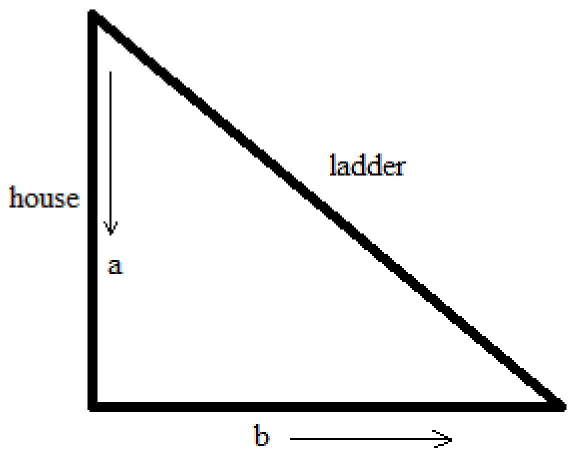
The slope of the normal is $m = \frac{-1}{2}$.

The equation of the normal line is $y - 1 = \frac{-1}{2}(x - 0)$ or $y = \frac{-1}{2}x + 1$.

8. A 13-foot ladder is leaning against the wall of a house. The base of the ladder slides away from the wall at a rate of 0.75 feet per second. How fast is the top of the ladder moving down the wall when the base is 12 feet from the wall?

G. $\frac{-5}{9}$ ft/sec

Perhaps a sketch will help:



Use the Pythagorean Theorem: $13^2 = a^2 + b^2$, where the length of the ladder is 13 feet.

We are also given that $\frac{db}{dt} = 0.75$ ft/sec, which indicates that the ladder slides away from the wall at this rate.

Now take the derivative and substitute for db/dt: $0 = 2a \bullet \frac{da}{dt} + 2b \bullet \frac{db}{dt}$, so $0 = a \bullet \frac{da}{dt} + b \bullet (0.75)$

We know that a = 5 when b = 12 (use the Pythagorean theorem), so we have:

$0 = 5 \bullet \frac{da}{dt} + 12 \bullet (0.75)$ or $\frac{da}{dt} = -\frac{5}{9}$ ft/sec

Note that the answer is negative to represent the ladder sliding down the wall.

9. Find the intervals on which $f(x)=\dfrac{x^2}{x^2-4}$ is increasing. T. $(-\infty,-2)$ and $(-2,0)$

Find the derivative using the quotient rule:

$$f'(x)=\frac{\left(x^2-4\right)(2x)-\left(x^2\right)(2x)}{\left(x^2-4\right)^2}=\frac{-8x}{\left(x^2-4\right)^2}$$

Setting the numerator equal to 0 gives critical points where the derivative equals 0 (at x = 0), and setting the denominator equal to 0 gives critical points where the derivative is undefined (at x = – 2 at x = 2). Note that – 2 and 2 are not critical points because they are not in the domain of the original function. However, the graph can change from increasing to decreasing and vice versa around these values, so they need to be considered.

Use either the number line approach or the graph of the first derivative to conclude that the function is increasing on $(-\infty,-2)$ and $(-2,0)$.

10 – 11.

10. Find the absolute maximum for $f(x) = x^3 – 4x^2 + 1$ on [–1, 5]. A. 26

Take the derivative and set it equal to 0 to find critical points.

$f'(x) = 3x^2 – 8x$

$3x^2 – 8x = 0$, so $x(3x – 8) = 0$, and therefore, x = 0 and 8/3.

Evaluating *f* at each critical point gives:

$f(0) = 0)^3 – 4(0)^2 + 1 = 1$; $f\left(\frac{8}{3}\right)=\left(\frac{8}{3}\right)^3-4\left(\frac{8}{3}\right)^2+1=\frac{-229}{27}$

Evaluating *f* at each endpoint gives:

$f(–1) = (– 1)^3 – 4(–1)^2 = –4$; $f(5) = (5)^3 -4(5)^2 + 1 = 26$

The absolute maximum of 26 occurs when x = 5.

11. Find the absolute minimum for $f(x) = x^3 – 4x^2 + 1$ on [–1, 5]. U. $\dfrac{-229}{27}$

From the discussion in #10, you can see that the absolute minimum of $\dfrac{-229}{27}$ occurs when $x=\dfrac{8}{3}$.

12. Determine the slope of $9x - 4x\ln y = 3$ at $\left(\frac{1}{3}, 1\right)$. I. $\frac{27}{4}$

The slope of a curve is found by taking its derivative.

$$9x - 4x\ln y = 3$$

$$9 - \left(4x\left(\frac{y'}{y}\right) + 4\ln y\right) = 0$$

$$9 - \frac{4xy'}{y} + 4\ln y = 0$$

$$\frac{-4xy'}{y} = 4\ln y - 9$$

$$-4xy' = 4y\ln y - 9y$$

$$y' = \frac{4y\ln y - 9y}{-4x}$$

$$y'\left(\frac{1}{3}, 1\right) = \frac{4(1)\ln(1) - 9(1)}{-4\left(\frac{1}{3}\right)}$$

$$y'\left(\frac{1}{3}, 1\right) = \frac{4 \bullet 0 - 9}{\frac{-4}{3}} = \frac{0 - 9}{\frac{-4}{3}} = \frac{27}{4}$$

13. Determine the points of inflection for the function $f(x) = 4x^3 + 6x^2 - 5$. H. $\left(\frac{-1}{2}, -4\right)$

$$f'(x) = 12x^2 + 12x$$

$$f''(x) = 24x + 12$$

Set second derivative equal to 0 to check for points of inflection.

$$f''(x) = 24x + 12 = 0$$

$$24x = -12$$

$$x = \frac{-1}{2}$$

Now check the second derivative for a point on either side of $x = \frac{-1}{2}$.

$f''(-1) = -12$ so it is concave down at x = -1 and

$f''(0) = 12$ so it is concave up at x = 0.

Therefore, there is an inflection point at (-1/2, -4).

14. Determine the y-intercept of the line passing through the point (– 5, 4) and perpendicular to the line 4x – 3y = 5.

P. $\frac{1}{4}$

4x - 3y = 5 $\rightarrow$ -3y=-4x+5 $\rightarrow$ $y=\left(\frac{4}{3}\right)x-\left(\frac{5}{3}\right)$

So, the slope = $\left(\frac{4}{3}\right)$.

The slope of the perpendicular line = $\left(\frac{-3}{4}\right)$.

Since it passes through the point (-5,4), we can write:

$$y=mx+b$$

$$4=\left(\frac{-3}{4}\right)(-5)+b$$

$$4=\frac{15}{4}+b$$

$$16=15+4b$$

$$1=4b$$

$$b=\frac{1}{4}$$

15. Determine the interval over which the curve $y=\frac{x-1}{3+x}$ is concave down.

O. $(-3,\infty)$

$$y=\frac{x-1}{3+x}$$

$$y'=\frac{(3+x)-(x-1)}{(3+x)^2}=\frac{3+x-x+1}{(3+x)^2}=\frac{4}{(3+x)^2}=4(3+x)^{-2}$$

$$y''=-8(3+x)^{-3}$$

The only point you need to check is x = – 3.

$y''(-4)=8$, so it is concave up at $x=-4$.

$y''(0)=\frac{-8}{27}$, so it is concave down at $x=0$.

Therefore, the function is concave down on the interval $(-3,\infty)$.