The Mathematics of a Football Kick

(Parametric Equations)
(Adapted from an article in COMAP)

I. Problem

The punter on the 1994 Handley State Championship Football team, Michael Partlow, was also a calculus student.
The question I posed to him was "Is it possible to determine the velocity of the football as it leaves his foot on his best kicks?"

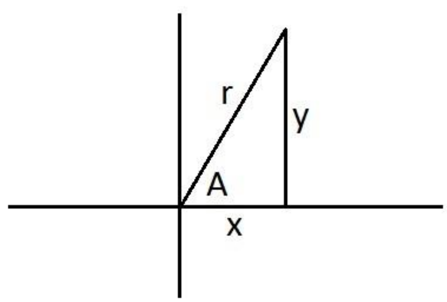
II. Background

Examine the formula $s = v_0 t - 16\,t^2$ which gives the height s of an object if it is thrown vertically up in the air with an initial velocity of v_0.
The units are feet (s) and seconds (t).

But a football is not kicked straight up (hopefully), but rather at an angle.
So we need to revisit some formulas from Precalculus or Physics. Looking at the diagram at the right, fill in the following:

cos A = ______

sin A = ______



so, **x = r cos A** and **y = r sin A** if there is no gravity.

Recall from algebra that **distance = velocity x time,** so, **r = v t**

Substituting for r in the formulas above, we get

x = ______________________________

y = ______________________________ **- 16 t^2**

These formulas neglect wind velocity. Note that only the vertical motion is affected by gravity - that is why the $-16t^2$ is only subtracted from the y-value.

III. Assumptions

We will assume that there is no wind velocity.

We will make the conjecture that the best angle at which to kick the football to achieve the maximum distance is a 45° angle (we will prove this later)

IV. Solving the Problem

Michael Partlow's best punt went 56 yards (which is equivalent to 168 feet).

Substituting the values in the two equations above, we get

$$\mathbf{168 = v\ t\cos 45°}$$
$$\mathbf{0 = v\ t\sin 45° - 16\ t^2}$$

Now solve these equations simultaneously for t and v.

SHOW WORK:

t = ________________ (which represents the "hang time")

v = ________________ (which represents the initial velocity)

V. Doing Parametric Equations on the TI-85 Graphing Calculator

A. You must first change some of the settings:

Press **MODE**

Select **NORMAL**

FLOAT

DEGREE

RECTC

PARAM

B. To enter the data for the problem above:

1. Type the following:

45 STO A

73.329 STO V

2. Press **GRAPH**

   Press **RANGE**

   tMin = 0
   tMax = 5
   tStep = .05
   ----
   xMin = 0
   xMax = 200
   xScl = 10
   ----
   yMin = -20
   yMax = 60
   yScl = 10

3. Press **E(t)=**

   xt1 **= V t cos A**
   yt1 = **V t sin A - 16t^2**

   Press **GRAPH**

4. Use the **TRACE** button to find the following:

   a. Find x when y = 0: **x = ______________**

   b. Find the values of x, y, and t at its highest point:
   x = ___________ y = ___________ t = __________

VI. Using calculus to solve for the information above:

Determine the time that it takes for the ball to reach its highest point.

Take the formula **$y = V\,t\,\sin A - 16t^2$** and substitute 73.329 in for **V** and 45° for **A**.
Then solve for **dy/dt** and set it equal to 0 (WHY???)
Solve this equation for **t** (this will give you half of the "hang time."
SHOW ALL WORK:

VII. To show that 45° is the optimal angle for kicking a football the farthest distance:

1. First determine a general equation for **x** when y = 0.

Begin with the parametric equations

$y = V\,t \sin A - 16t^2$ and

$x = V\,t \cos A$

Set y = 0 to obtain **$0 = V\,t \sin A - 16t^2$**

Solve for **t (SHOW WORK):**

Substitute the nonzero value of **t** in the parametric equation **$x = V\,t \cos A$**

x = ___________________________________

Now use the double angle identity (**sin2A = 2sinA cosA)**

to express x in terms of sin 2A:

x = ____________________________________

This says that if V is constant, the distance (x) varies sinusoidally with the angle.

2. Use calculus to solve for the angle which gives the best distance:

Take the equation $x = \frac{V^2}{32}\sin(2A)$ and find $\frac{dx}{dA}$

(Remember **V** is a constant)

Then set dx/dA = 0 to get the maximum distance (why?) and solve for **A**:

SHOW WORK:

3. Use a calculator to solve for the optimal angle:

First, select **MODE** and change back to **FUNC**

Press **GRAPH** and select **y(x)=**

Graph $x = \frac{V^2}{32}\sin(2A)$ by entering **y1 = (73.329)^2 sin (2x) / 32**

Select **RANGE** and enter the following:

xMin= 0
xMax= 90
xScl= 5
yMin= 0
yMax= 200
yScl= 10

Please note that on the calculator, the variable **x** represents the angle, and the variable **y** represents the distance the ball is kicked (the **x** value in the formulas above)

Use the **TRACE** function to determine the optimal angle to kick the ball.

Give these values of **x** and **y**: **x** = ______________

y = ______________

Also give the following values:

When x = 50° y = ____________

When x = 30° y = ____________

VIII. Additional Explorations:

A. We can modify our parametric equations to **x = V t cos A + W t** and **y = V t sin A + H** where **W** is the velocity of a wind blowing with (+) or against (-) the kicker, and **H** is the height of the ball when it is kicked.

How much is the problem affected if the ball is kicked from one foot off the ground?

B. How does the wind affect the kick? For example, if there is a wind of 10 ft/sec with the kicker, at what angle should you kick the ball to maximize the distance?

The equation is $\mathbf{x = (V^2 \sin 2A) / 32 + (10V \sin A) / 16}$

C. Suppose you are kicking a field goal. In this case, the ball must be more than ten feet above the ground to get over the goal posts. How much distance is lost to achieve this condition?

D. Tom Dempsey holds the NFL record for kicking the longest field goal (63 Yards). With what velocity must he have kicked the ball if his angle was optimal?

E. How quickly does the ball go up? Suppose a player with his arms raised can block a punt if it is under seven feet high. How close to the kicker must he get? What if the angle of the kick is 55°?