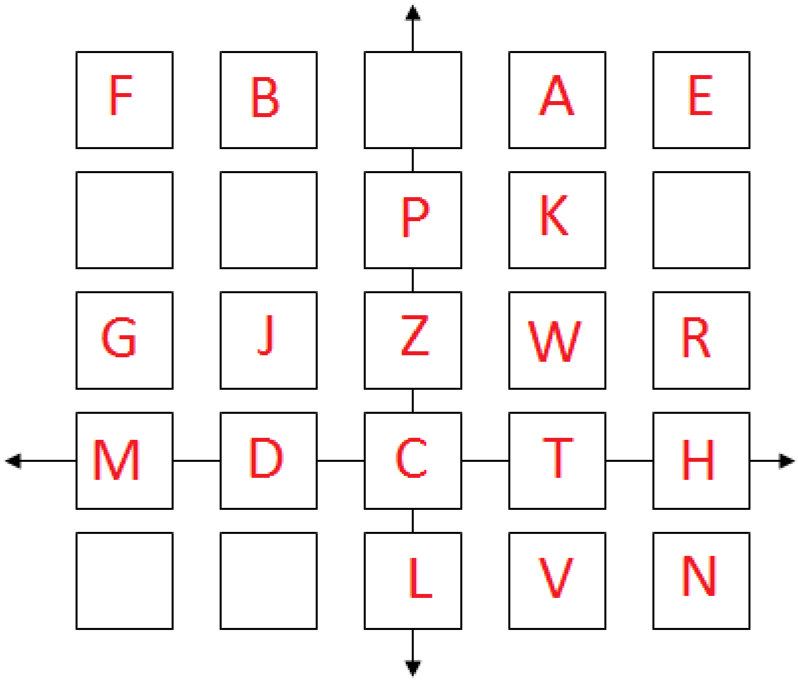
Solution Key **Class, Take Your Seats #4** *by David Pleacher* Equations of Tangents and Normals



CLUES:

2. Abel is seated on the line which is normal to the curve $f(x) = x^2 - 2x + 4$ at the point (1, 3).

   Take the derivative to find the slope of the tangent: y' = 2x – 2.
   Then substitute x = 1 into the derivative to get 0. So the slope of the normal must be undefined. Therefore the normal line is x = 1.
   And there are five points in our domain that satisfy that condition:
   (1, -1), (1, 0), (1, 1), (1, 2), and (1, 3).

3. Brahmagupta sits on the line normal to the curve $y = x^5 - x^4 + 1$ at x = 1.

   First find the y-coordinate: $y(2) = 2^5 - 2^4 + 1 = 17$.
   Then take the derivative: $y' = 5x^4 - 4x^3$
   Find the slope at x = 1: y'(1) = 5 – 4 = 1.
   Now, take the negative reciprocal to get the slope of the tangent: m = -1
   Finally, the equation becomes y = -x – 1.
   And there are four points which lie on this line:
   (-1, 3), (0, 2), (1, 1), and (2, -1).

4. Cantor is located on the line tangent to the curve $y = -x^2 + 10x - 25$ at the point (5, 0).

   First, the derivative y' = -2x + 10 to find the slope of the tangent.
   Substituting x = 5, we obtain a slope of zero. So, the tangent line is the x-axis or y = 0.
   So, Cantor sits at (-1, 0), (-2, 0), (0, 0), (1, 0), or (2, 0).

5. Descartes is seated on the line normal to $y = -x - x^2$ at $x = -1$.

First take the derivative: dy/dx = -1 – 2x.
The slope of the tangent is 1.
The y-coordinate is 0.
The slope of the normal is -1.
The equation of the normal is y – 0 = -1 (x + 1) or y = -x – 1
There are three points which lie on this line:
(-2, 1), (-1, 0), and (0, -1).

6. Euclid sits on the line tangent to $y = x^3 + x^2$ at (3, 36).

The derivative is: dy/dx = $3x^2$ + 2x.
The slope of the tangent line is 33.
The equation is (y – 36) = 33 (x – 3) or y = 33x – 63.
The only point in our domain that falls on this line is (2, 3), so Euclid sits at (2, 3).

7. Fermat is located on the line tangent to $y = \sqrt{x^2 + 5}$ at the point $(-2, 3)$.

First, take the derivative of the equation: $\frac{dy}{dx} = \frac{1}{2}\left(x^2 + 5\right)^{-\frac{1}{2}}(2x)$

Now, plug in x = - 2 to get the slope of the tangent line: $\frac{dy}{dx} = \frac{1}{2}(9)^{-\frac{1}{2}}(-4) = \frac{-2}{3}$

Third, plug the slope and the point into the equation for the line:

$y - 3 = \frac{-2}{3}(x + 2)$ which simplifies to 2x + 3y = 5

The only points in our domain which falls on this line are the points (1, 1) and (-2, 3).

8. The curve $y = ax^2 + bx + c$ passes through the point (2, 4) and is tangent to the line y = x + 1 at (0, 1). Determine values for a, b, and c. Gauss sits at the point $(-b - c, 4a)$.

The curve passes through (2, 4), so if you plug in x = 2, you'll get y = 4.
Therefore, 4 = 4a + 2b + c
Second, the curve also passes through the point (0, 1), so c = 1.
Because the curve is tangent to the line y = x + 1 at (0, 1), they must both have the same slope at that point. The slope of the line is 1. The slope of the curve is the first derivative: $\frac{dy}{dx} = 2ax + b$, and at (0, 1) $\frac{dy}{dx} = b$. Therefore, b = 1.
Now that you know b and c, plug them back into the equation and solve for a:

4 = 4a + 2 + 1, and $a = \frac{1}{4}$.

Therefore, Gauss must sit at the point (-1 – 1, 4(1/4)), or at (-2, 1).

9. Hardy sits at one of the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

The x-axis is a horizontal line, so its slope is zero, Therefore, you want to know where the derivative of this curve is zero.
Take the derivative and set it equal to zero:

$$\frac{dy}{dx} = 6x^2 - 6x - 12, \quad \text{so} \quad 6x^2 - 6x - 12 = 0$$

$$6(x^2 - x - 2) = 0$$

$$6(x-2)(x+1) = 0$$

$$x = 2 \text{ or } x = -1$$

Only x = 2 yields a point in our range, so Hardy must sit at the point (2, 0).

10. Jacobi is seated on the line tangent to the graph of $y = 2x^3 - 3x^2 - 12x + 21$ at x =2.

Find the y-coordinate when x = 2: y = 16 – 12 – 24 +21 = 1
Find the derivative of the curve to get the slope of the tangent: $y' = 6x^2 - 6x - 12$
Plug in x = 2 into the derivative: $y'(2) = 6(2)^2 - 6(2) - 12 = 0$
So, the equation of the tangent is: y – 1 = 0 (x – 2) or y = 1.
Therefore, Jacobi could sit at one of the five points: (-2, 1), (-1, 1), (0, 1), (1, 1), (2, 1)

11. Klein is located on the tangent line to $y = 3x^2 - x$ at x = 1.

When x = 1, y = 2.
The derivative y'=6x – 1.
So, the slope of the tangent is 6(1) – 1 = 5.
The equation of the tangent is y – 2 = 5 (x – 1) or y = 5x – 3.
The only point in our domain that falls on this line is (1, 2).

12. Leibniz sits on the line which is tangent to the curve $y = 4x^2 - 22x + 35$ at the point (3, 5).

The tangent to a curve has the same slope as the curve at the point of tangency.
So take the first derivative: y' = 8x - 22
The slope at x = 3 is m = 24 - 22 = 2
Now use the slope and point to find the equation of the tangent:
y - 5 = 2 (x - 3)
So, Leibniz sits on the line y = 2x - 1, and the points
in our domain that fall on this line are (1, 1), (0, -1), and (2, 3).

13. Mandelbrot sits at the point on the curve $y = (x+2)^2$ where the normal to that curve is parallel to the y-axis.

The graph of $y = (x+2)^2$ is a parabola opening up with its vertex at (-2, 0). The tangent line is the x-axis or y = 0. So the normal line is x = -2. (Calculus could also be used).
Therefore, Mandelbrot must sit at the point (-2, 0).

14. Newton's seat is collinear with those of Gauss and Cantor.

At this point, Cantor must sit at (-1, 0), (0, 0), or (1, 0) and we know Gauss is at (-2, 1).
So, Newton must sit at either (0, -1) or ((2, -1) to be collinear with Gauss and Cantor.

15. Determine the values of a, b, and c where the curves $y = x^2 + ax + b$ and $y = cx + x^2$ have a common tangent line at $(-1, 0)$. Pythagoras sits at the point (b, a+c).

If the curves have a common tangent at (-1, 0), then their slopes at that point must be equal.
Take the derivatives and plug in x = -1.
For $y = x^2 + ax + b$, dy/dx = 2x + a, so the slope = -2 + a at (-1, 0).
For $y = cx + x^2$, dy/dx = c + 2x, so its slope at (-1, 0) is c – 2.
So, -2 + a = c – 2. (equation #1)
Since the two curves are tangent at (-1, 0), substitute x = -1 and y = 0 into both equations:
For $y = x^2 + ax + b$, $0 = (-1)^2 + a(-1) + b$ or 0 = 1 –a + b (equation #2)
For $y = cx + x^2$, $0 = c(-1) + (-1)^2$ or 0 = -c + 1 (equation #3)
Since we have three equations with three unknowns, we can solve:
In equation #3, c = 1.
Then substitute this value of c in equation #1 to get a = 1.
Now substitute for a in equation #2 to get b = 0.
Hence, Pythagoras sits at the point (0, 2).

16. Riemann sits on the line normal to the curve $y = x^2 - 3x + 2$ at x = 1.

First, determine the y-coordinate when x = 1: y = 0.
Next, take the derivative to find the slope of the tangent at (1, 0):
dy/dx = 2x – 3, so the slope of the tangent at (1, 0) is -1.
So, the slope of the normal is m = +1.
The equation of the normal is y – 0 = 1 (x – 1) or y = x – 1.
Therefore, Riemann sits at either (0, -1), (1, 0), or (2, 1).

17. The line tangent to a curve at a point (x_1, y_1) is $y = 2x - 2$. The normal to that curve at the same point passes through (11, -5). Taylor sits at the point (x_1, y_1).

The slope of the tangent line is m = 2 (using y = mx + b or the derivative of $y = 2x - 2$).
So, the slope of the normal is $\frac{-1}{2}$.

The equation of the normal is $(y - (-5)) = \frac{-1}{2}(x - 11)$, which simplifies to $y = \frac{-1}{2}x + \frac{1}{2}$ or $x + 2y = 1$.

Now solve this equation with the tangent equation to find the point of intersection (x_1, y_1).
Taylor must sit at this point of intersection (1, 0).

18. Venn's seat is collinear with those of Brahmagupta and Zeno.

At this point, we have not solved for Zeno, so come back to this clue.

19. Wallis is seated on the line tangent to $y = 4 - 3x - x^2$ at the point (2, -6).

First, take the derivative of the equation : $\frac{dy}{dx} = -3 - 2x$

Now, plug in x = 2 to get the slope of the tangent line: $\frac{dy}{dx} = -3 - 2(2) = -7$

Third, plug the slope and the point into the equation for the line:

$y - (-6) = -7(x - 2)$ which simplifies to $y = -7x + 8$.

The only point in our domain which satisfies this equation is (1, 1), so Wallis sits here.
But that means that Fermat must sit at (-2, 3) and Brahmagupta must be at (-1, 3).
Also, Leibniz must be at (0, -1), which forces Newton to be at (2, -1).
That tells us that Cantor must sit at (0, 0) and it follows that Descartes must be at (-1, 0).

20. Zeno is located on the line tangent to $y=\frac{2x+5}{x^2-3}$ at x = 1.

First, find the y-coordinate: $y(1)=\frac{2(1)+5}{1^2-3}=-\frac{7}{2}$

Second, take the derivative: $\frac{dy}{dx}=\frac{\left(x^2-3\right)\left(2\right)-\left(2x+5\right)\left(2x\right)}{\left(x^2-3\right)^2}$

Don't bother simplifying the first derivative – just plug in x = 1:

$$\frac{dy}{dx}=\frac{\left(1^2-3\right)\left(2\right)-\left(2(1)+5\right)\left(2(1)\right)}{\left(x^2-3\right)^2}=\frac{-4-14}{4}=-\frac{9}{2}$$

Now, we have the slope and a point, so the equation is:

$$y+\frac{7}{2}=-\frac{9}{2}\left(x-1\right) \text{ or } 2y = -9x + 2.$$

The only point in our domain that satisfies this equation is the point (0, 1),
So Zeno sits at (0. 1).
Now go back to Clue #18, to see that Venn must sit at the point (1, -1).
This forces Abel to sit at (1, 3).
That means that Riemann must sit at (2, 1) and that leaves Jacobi sitting at (-1, 1).

So, the order in which you can determine the seats for the nineteen students is:

1. E
2. G
3. H
4. K
5. M
6. P
7. T
8. W
9. F
10. B
11. L
12. N
13. C
14. D
15. Z
16. V
17. A
18. R
19. J