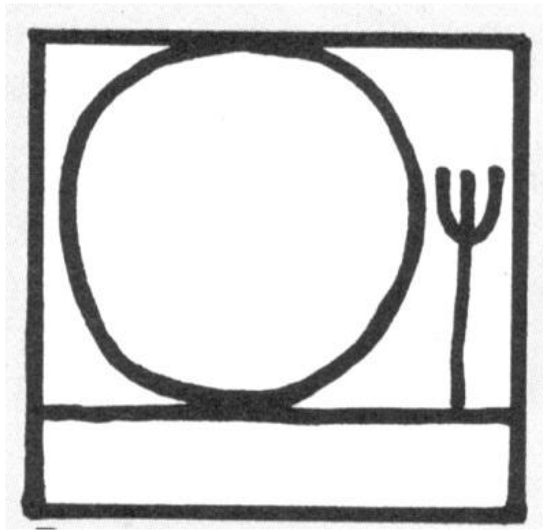
Turvy for Limits & Continuity-- A Puzzle by David Pleacher

SOLUTION KEY



Here is the title right-side-up:

D E V I L   R U N   O V E R   B Y   A

6 10 12 16 17   14 3 13   19 12 10 14   8 2   21

G I A N T   S N O W B A L L

15 16 21 13 11   20 13 19 1 8 21 17 17

Here is the title upside-down:

O A T M E A L   C O O K I E

19 21 11 7 10 21 17   9 19 19 4 16 10

G R A D U A T I N G   F R O M

15 14 21 6 3 21 11 16 13 15   5 14 19 7

H I G H   S C H O O L

18 16 15 18   20 9 18 19 19 17

___ 1. If $c \neq 0$, evaluate $\lim_{x \to c} \frac{x^3 - c^3}{x^6 - c^6}$

Rewrite $\lim_{x \to c} \frac{x^3 - c^3}{x^6 - c^6}$ as

$$\lim_{x \to c} \frac{x^3 - c^3}{\left(x^3 - c^3\right)\left(x^3 + c^3\right)} = \lim_{x \to c} \frac{1}{\left(x^3 + c^3\right)}$$

Substituting $x = c$, you obtain $\frac{1}{c^3 + c^3} = \frac{1}{2c^3}$

___ 2. $\lim_{x \to 0} (x - 5)\cos(x) =$

Using the product rule,
$\lim_{x \to 0} (x - 5)(\cos x) =$

$\left[\lim_{x \to 0} (x - 5)\right] \left[\lim_{x \to 0} (\cos x)\right]$

$= (0 - 5)(\cos 0) = (-5)(1) = -5.$
(Note that $\cos 0 = 1$.)

___ 3. Evaluate $\lim_{x \to 0} \frac{2 - \sqrt{4 - x}}{x}$

Substituting $x = 0$ into the expression $\frac{2 - \sqrt{4 - x}}{x}$ leads to $0/0$ which is an indeterminate form. Thus, multiply both the numerator and denominator by the conjugate $\left(2 + \sqrt{4 - x}\right)$ and obtain

$$\lim_{x \to 0} \frac{2 - \sqrt{4 - x}}{x} \left(\frac{2 + \sqrt{4 - x}}{2 + \sqrt{4 - x}}\right)$$

$$= \lim_{x \to 0} \frac{4 - (4 - x)}{x\left(2 + \sqrt{4 - x}\right)}$$

$$= \lim_{x \to 0} \frac{x}{x\left(2 + \sqrt{4 - x}\right)}$$

$$= \lim_{x \to 0} \frac{1}{\left(2 + \sqrt{4 - x}\right)}$$

$$= \frac{1}{\left(2 + \sqrt{4 - (0)}\right)} = \frac{1}{4}.$$

___ 4. Evaluate $\lim_{x\to\infty}\frac{5-6x}{2x+13}$

Since the degree of the polynomial in the numerator is the same as the degree of the polynomial in the denominator,

$$\lim_{x\to\infty}\frac{5-6x}{2x+13}=\frac{-6}{2}=-3$$

___ 5. Evaluate $\lim_{x\to\infty}\frac{5x^2-6x+9}{x^3-2x^2}$

Since the degree of the polynomial in the numerator is 2 and the degree of the polynomial in the denominator is 3,

$$\lim_{x\to-\infty}\frac{x^2+2x-3}{x^3+2x^2}=0.$$

___ 6. Determine the value of k that makes the function f(x) continuous on [0, 11], if f(x) is defined as follows:

$$f(x)=\begin{cases} k\bullet\sin\frac{(x+3)\pi}{6}, & x\le 2\\ \frac{3-\sqrt{11-x}}{x-2}, & x>2\end{cases}$$

In order for *f* to be continuous, it can't have a break in the graph when x = 2. Therefore, you have to get the same output from both pieces of the function if you plug in 2 for x.

Let's start with x >2:

you can't just plug in 2 for x because you get an indeterminate answer (0 divided by 0), so instead, you must calculate the limit of that function as x approaches 2 from the right side using the conjugate method of finding limits:

$$\lim_{x\to2}\frac{3-\sqrt{11-x}}{x-2}\bullet\frac{3+\sqrt{11-x}}{3+\sqrt{11-x}}$$

$$=\lim_{x\to2}\frac{9-(11-x)}{(x-2)\left(3+\sqrt{11-x}\right)}$$

$$=\lim_{x\to2}\frac{(x-2)}{(x-2)\left(3+\sqrt{11-x}\right)}$$

$$=\lim_{x\to2}\frac{1}{\left(3+\sqrt{11-x}\right)}$$

$$=\frac{1}{\left(3+\sqrt{11-2}\right)}=\frac{1}{6}$$

This tells us that the other function, with 2 plugged in for x must also equal 1/6, so write that as an equation and solve for k:

$$k \bullet \sin\frac{(2+3)\pi}{6} = \frac{1}{6}$$
$$k \bullet \sin\frac{5\pi}{6} = \frac{1}{6}$$
$$k \bullet \frac{1}{2} = \frac{1}{6}$$
$$k = \frac{1}{3}$$

7 – 8.

Given $f(x) = \begin{cases} \ln x & \text{if } 0 < x < 1 \\ ax^2 + b & \text{if } 1 \le x < \infty \end{cases}$

If $f(2)$ =3, determine the values of a and b for which f(x) is continuous on the interval $(0,\infty)$.

If f(x) is to be continuous, f(1) must have the same value from the right side and from the left side.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \ln x = \ln(1) = 0$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ax^2 + b = a + b$$

So, a + b = 0, therefore a = -b.

Now substitute f(2) = 3 into the second expression to get 3 = 4a + b.

Solve these equations simultaneously to get 3 = 3a or a = 1. Then b = -1.

___ 9. Evaluate $\lim_{x\to\infty} \frac{3x^3+9}{5x+8}$

The degree of the monomial in the numerator is 2 and the degree of the binomial in the denominator is 1. Thus,

$$\lim_{x\to\infty} \frac{3x^3+9}{5x+8} = \infty$$

___ 10. Evaluate $\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2-4}}$

Divide every term in both the numerator and denominator by the highest power of *x*. In this case, it is *x*. Thus, you have

$$\lim_{x\to-\infty} \frac{\frac{3x}{x}}{\frac{\sqrt{x^2-4}}{x}}.$$

As $x \to -\infty$, $x = -\sqrt{x^2}$. Since the denominator involves a radical, rewrite the expression as

$$\lim_{x\to-\infty} \frac{\frac{3x}{x}}{\frac{\sqrt{x^2-4}}{-\sqrt{x^2}}} = \lim_{x\to-\infty} \frac{3}{-\sqrt{1-\frac{4}{x^2}}}$$

$$= \frac{3}{-\sqrt{1-0}} = -3$$

___ 11. If $f(x) = \begin{cases} e^x & \text{for } 0 \le x < 1 \\ x^2 e^x & \text{for } 1 \le x < 5 \end{cases}$

determine $\lim_{x\to 1} f(x)$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} \left(x^2e^x\right) = e$ and $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (e^x) = e$. Thus, $\lim_{x\to1} f(x) = e$.

___ 12. Evaluate $\lim_{x\to\infty} \frac{e^x}{1-x^3}$

$\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} \left(1-x^3\right) = \infty$. However, as $x \to \infty$, the rate of increase of e^x is much greater than the rate of decrease of $(1-x^3)$. Thus, $\lim_{x\to\infty} \frac{e^x}{1-x^3} = -\infty$.

___ 13. Evaluate $\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}$

Divide both numerator and denominator by x and obtain $\lim_{x\to 0}\frac{\frac{\sin 3x}{x}}{\frac{\sin 4x}{x}}$. Now rewrite the limit as $\lim_{x\to 0}\frac{3\frac{\sin 3x}{3x}}{4\frac{\sin 4x}{4x}} = \frac{3}{4}\lim_{x\to 0}\frac{\frac{\sin 3x}{3x}}{\frac{\sin 4x}{4x}}$.

As x approaches 0, so do $3x$ and $4x$.

Thus, you have

$$\frac{3}{4}\frac{\lim_{3x\to 0}\frac{\sin 3x}{3x}}{\lim_{4x\to 0}\frac{\sin 4x}{4x}} = \frac{3(1)}{4(1)} = \frac{3}{4}.$$

___ 14. Given the function: $f(x)=\begin{cases}\frac{x^2-9}{x-3} & \text{for } x\neq 3\\ a & \text{for } x=3\end{cases}$

Determine the value of *a* which makes the function continuous.

In order to be continuous, the two expressions for f(x) must be equal.

$f(3)=a$ and $\lim_{x\to 3}\frac{x^2-9}{x-3}=\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6$

Therefore, a = 6.

15 – 16. Given the function: $f(x)=\begin{cases}\sin x & \text{if } x\leq -\frac{\pi}{2}\\ a\sin x+b & \text{if } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ 2\cos x & \text{if } x\geq \frac{\pi}{2}\end{cases}$

Determine the values of *a* and *b* so that the function *f*(x) is continuous for all values of x.

The points which must be examined are x = $-\frac{\pi}{2}$ and x = $\frac{\pi}{2}$.

$$\sin\left(-\frac{\pi}{2}\right)=a\sin\left(-\frac{\pi}{2}\right)+b \text{ and } a\sin\left(\frac{\pi}{2}\right)+b = 2\cos\left(\frac{\pi}{2}\right)$$

$$-1=-a+b \text{ and } a+b=0$$

Solving simultaneously, $b=-\frac{1}{2}$ and $a=\frac{1}{2}$

___ 17. Determine the points of discontinuity of the function $f(x)=\dfrac{1}{x^3-3x^2-x+3}$

The points of discontinuity occur when the denominator is equal to zero.

Set $x^3-3x^2-x+3=0$

and solve for x by factoring:

$$x^3-3x^2-x+3=0$$

$$x^2(x-3)-(x-3)=0$$

$$(x^2-1)(x-3)=0$$

$$(x-1)(x+1)(x-3)=0$$

So, $x=-1,1,3$

___ 18. Given the function: $f(x)=\begin{cases}|3\text{-x}| & \text{if } x<7\\ ax-10 & \text{if } 7\le x<10\end{cases}$

Determine the value of *a* so that the function f(x) is continuous on the interval $(-\infty,10)$.

In order to be continuous, the limit of f(x) as it approaches 7 from both the right side and the left side must be equal.

$$\lim_{x\to 7^-} f(x)=\lim_{x\to 7^-}|3\text{-x}|=|-4|=4$$

$$\lim_{x\to 7^+} f(x)=\lim_{x\to 7^+}(ax-10)=7x-10$$

Setting these two expressions equal:

$$7x-10=4$$

$$7x=14$$

$$x=2$$

___ 19. Evaluate $\lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

This is the definition of the derivative and is a good problem to give students before they encounter the derivative.

$$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}$$

$$=\lim_{h\to 0}\frac{h(2x+h)}{h}=\lim_{h\to 0}(2x+h)=2x$$

___ 20. Given the function: $f(x)=\begin{cases} |18\text{-x}| & \text{if } x<7 \\ x-10 & \text{if } x\geq 7 \end{cases}$

Evaluate $\lim_{x\to 7} f(x)$

If the limit exists, the limits from the left and right sides must be equal.

$\lim_{x\to 7^-} f(x)=\lim_{x\to 7^-} |18\text{-x}|=11.$

$\lim_{x\to 7^+} f(x)=\lim_{x\to 7^+} (x-10)=-4.$

Since the values are different, the limit doe not exist.

___ 21. Evaluate $\lim_{x\to 3}\frac{x^3-27}{x^2-9}$

$$\lim_{x\to 3}\frac{x^3-27}{x^2-9}=\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)}=\lim_{x\to 3}\frac{(x^2+3x+9)}{(x+3)}=\frac{9+9+9}{6}=\frac{9}{2}$$