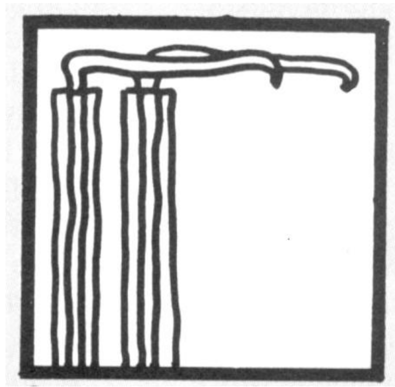
Turvy with Applications of the Integral -- Solution Key by David Pleacher



Here is the title right-side-up: Two candles in a hurricane

Here is the title upside-down: Uncle Sam wearing elf shoes

1. $\begin{cases}\text{Find the area in square units bounded by the curves} \\ y = x^3 - 2x^2 \text{ and } y = 2x^2 - x^3.\end{cases}$

D. $\frac{8}{3}$

Find the intersections of the curves:

$$x^3 - 2x^2 = 2x^2 - x^3$$

$$2x^3 - 4x^2 = 0$$

$$2x^2(x-2) = 0$$

$$x = 0,\ 2$$

So, the area $= \int_0^2 \left(\left(2x^2 - x^3\right) - \left(x^3 - 2x^2\right)\right) dx$

$$= \int_0^2 \left(4x^2 - 2x^3\right) dx = \left[\frac{4}{3}x^3 - \frac{1}{2}x^4\right]_0^2$$

$$= \frac{32}{3} - 8 = \frac{8}{3} \text{ square units}$$

2. $\begin{cases}\text{Using your calculator, determine the area of a region} \\ \text{bounded by the curves } y=\sin x,\ y=3x, \text{ and } y=30-3x.\end{cases}$

Intersections are at $x = 0, 5,$ and 10.243402.

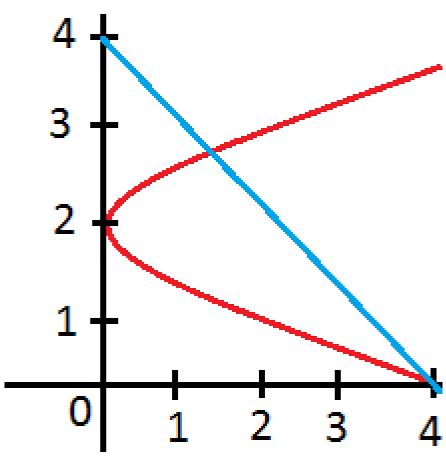
$$\text{Area } = \int_0^5 (3x-\sin x)\,dx + \int_5^{10.243402} (30-3x-\sin x)\,dx \approx 73.228 \text{ sq. units}$$

3. $\begin{cases}\text{Determine the area of the region bounded} \\ \text{by } x=(y-2)^2, \text{ and } y=4-x.\end{cases}$

Find the intersection of $x=(y-2)^2$ and $y=4-x$.

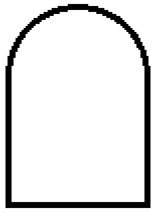
$$(y-2)^2 = 4-y$$
$$y^2-4y+4=4-y$$
$$y^2-3y=0$$
$$y=0,3$$

Draw a rough sketch or note that $4-y$ is larger than $(y-2)^2$ when $0<y<3$.

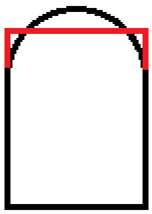


$$\text{Area } = \int_0^3 \left((4-y)-(y-2)^2\right)dy = \int_0^3 \left((4-y)-\left(y^2-4y+4\right)\right)dy$$
$$=\int_0^3 \left(3y-y^2\right)dy = \left[\frac{3}{2}y^2-\frac{1}{3}y^3\right]_0^3 = \frac{9}{2} \text{ square units}$$

4. $\begin{cases}\text{The figure below is a square of base 4 meters topped by} \\ \text{a semicircle. What is the average height of this figure?}\end{cases}$



You can think of a rectangle with this average height:



And realize that it will have the same area as the original figure.

The area of the original figure is the sum of the areas of the semicircle and square.

Area $= \frac{1}{2}\pi(2)^2 + 4^2 = 2\pi + 16.$

the area of the rectangle with average height is Area $= b \bullet \bar{h} = 4\bar{h}.$

Setting the two equal to each other, you find $4\bar{h} = 2\pi + 16.$

$\bar{h} = \frac{2\pi + 16}{4} = \frac{1}{2}\pi + 4$ meters

5. $\begin{cases}\text{Determine the area bounded by} \\ x = 2y^2 - 5 \text{ and } x = y^2 + 4.\end{cases}$

Determine the intersections of $x = 2y^2 - 5$ and $x = y^2 + 4,$

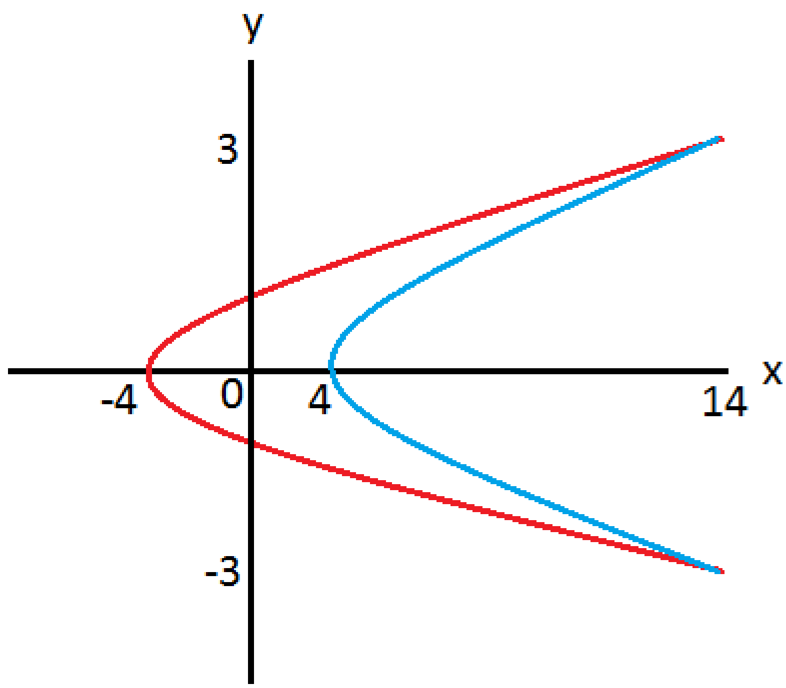
so $2y^2 - 5 = y^2 + 4.$

Intersections at $y = -3, 3.$

Then find the length of each differential rectangular element:

$L(y) = \left(y^2 + 4\right) - \left(2y^2 - 5\right) = 9 - y^2,$

since $y^2 + 4 \geq 2y^2 - 5$ when $-3 \leq y \leq 3.$



$$\text{Area} = \int_{-3}^{3}\left(\left(y^2+4\right)-\left(2y^2-5\right)\right)dy = \int_{-3}^{3}\left(9-y^2\right)dy$$

$$= \left[9y - \frac{1}{3}y^3\right]_{-3}^{3} = 36 \text{ square units.}$$

6. Determine the area bounded by $y = x$, $y = -\dfrac{x}{2}$ and $y = 5$.

The area can be cut vertically to give

$$A = \int_{-10}^{0}\left(5+\frac{x}{2}\right)dx + \int_{0}^{5}\left(5-x\right)dx = 25 + \frac{25}{2} = \frac{75}{2}$$

or it can be cut horizontally to give

$$A = \int_{0}^{5}\left(y+2y\right)dy = \frac{75}{2} \text{ square units.}$$

7. $\begin{cases} \text{Determine the area of the region bounded} \\ \text{by } y = \sin x,\ y = \csc^2 x,\ x = \dfrac{\pi}{4} \text{ and } x = \dfrac{3\pi}{4}. \end{cases}$

First draw a diagram and notice that the length of the rectangles to be summed is given by the distance between $\csc^2 x$ and $\sin x$.

So, the Area $= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\csc^2 x - \sin x\right) dx$

$$= \left[-\cot x + \cos x\right]_{\pi/4}^{3\pi/4} = \left(1 - \frac{\sqrt{2}}{2}\right) - \left(-1 + \frac{\sqrt{2}}{2}\right) = 2 - \sqrt{2} \text{ sq. units}$$

8. $\begin{cases} \text{Determine the area of the region IN THE FIRST} \\ \text{QUADRANT bounded by the curves by} \\ y = \sin x \cos^2 x,\ y = 2x\cos\left(x^2\right) \text{ and } y = 4 - 4x. \end{cases}$

You will need to use your calculator to find the intersections of the curves:

$y = 4 - 4x$ intersects the curve $y = \sin x \cos^2 x$ at $x = .928113.$

$y = 4 - 4x$ intersects the curve $y = 2x\cos\left(x^2\right)$ at $x = .692751.$

From your diagram, you will need to split up the area into two integrals and sum:

$$\text{Area } = \int_0^{.692751} \left(2x\cos\left(x^2\right) - \sin x \cos^2 x\right) dx + \int_{.692751}^{.928113} \left(4 - 4x - \sin x \cos^2 x\right) dx = .379 \text{ sq. units}$$

9. $\begin{cases} \text{Determine the number } a \text{ so that} \\ \int_2^5 x^2\, dx \text{ is the same as } \int_2^5 a\, dx. \end{cases}$

$$\int_2^5 x^2\, dx = 39$$

$$\int_2^5 a\, dx = ax\Big|_2^5 = a(5-2) = 3a$$

Now set $3a = 39$, so $a = 13$

10. A solid is formed by revolving around the x-axis the region bounded by the x-axis and the curve $y = \sqrt{\sin x}$ for $0 \le x \le \pi$. Determine the volume of the solid.

$$V = \int_0^{\pi} \pi \sin x \, dx = \left[-(\cos x)\pi\right]_0^{\pi} = 2\pi \text{ cubic units}$$

11. The acceleration function (in meters per second) and initial velocity are given for an object moving along a straight line:
$a(t) = 4t - 1, \quad v(0) = -6.$
Determine the total distance traveled by the object in the first 5 seconds.

First, you need to solve the differential equation to find the velocity:
Integrate $a(t) = 4t - 1$ to get $v(t) = 2t^2 - t + C$
Then use the initial condition $v(0) = -6$ to solve for $C = -6$
The velocity of the object is given by $v(t) = 2t^2 - t - 6.$

The total distance traveled by the object in the first five seconds is

$$s = \int_0^5 \left|2t^2 - t - 6\right| dt$$

Now, $2t^2 - t - 6$ has roots $t = 2$ and $t = \frac{-3}{2}.$

The function is negative for $\frac{-3}{2} < t < 2$ and positive for $t > 2.$

So, the total distance traveled by the object in the first 5 seconds is

$$\int_0^5 \left|2t^2 - t - 6\right| dt = -\int_0^2 \left(2t^2 - t - 6\right) dt + \int_2^5 \left(2t^2 - t - 6\right) dt$$

$$= -\left[\frac{2}{3}t^3 - \frac{1}{2}t^2 - 6t\right]_0^2 + \left[\frac{2}{3}t^3 - \frac{1}{2}t^2 - 6t\right]_2^5 = \frac{26}{3} + \frac{99}{2} = \frac{349}{6} \text{ meters}$$

12. $\begin{cases}\text{Determine the volume of the solid that results when} \\ \text{the region between the curve } y = x \text{ and the x-axis,} \\ \text{from } x = 0 \text{ to } x = 1, \text{ is revolved around the x-axis.}\end{cases}$

Sketch the diagram and slice vertically.

$$V = \pi \int_0^1 x^2 \, dx = \pi \left. \frac{x^3}{3} \right|_0^1 = \frac{\pi}{3} \text{ cubic units}$$

13. $\begin{cases}\text{Determine the volume of the solid that results when} \\ \text{the region bounded by } y = x \text{ and } y = x^2, \text{ from } x = 0 \\ \text{to } x = 1, \text{ is revolved about the x-axis.}\end{cases}$

First, sketch the curves. The top curve is $y = x$ and the bottom curve is $y = x^2$ throughout the region. So, the volume is

$$V = \pi \int_0^1 x^2 \, dx - \pi \int_0^1 x^4 \, dx = \pi \left. \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \right|_0^1 = \frac{2\pi}{15} \text{ cubic units}$$

14. $\begin{cases}\text{Determine the volume of the solid that results when} \\ \text{the region bounded by } x = y^2 \text{ and } x = y^3, \text{ from } y = 0 \\ \text{to } y = 1, \text{ is revolved about the y-axis.}\end{cases}$

First, sketch the curves and note that $x = y^2$ is always on the outside and $x = y^3$ is always on the inside.

So to find the volume, you must evaluate the integral:

$$V = \int_0^1 \pi \left(y^4 - y^6 \right) dy = \pi \left[\frac{y^5}{5} - \frac{y^7}{7} \right]_0^1 = \frac{2\pi}{35} \text{ cubic units.}$$

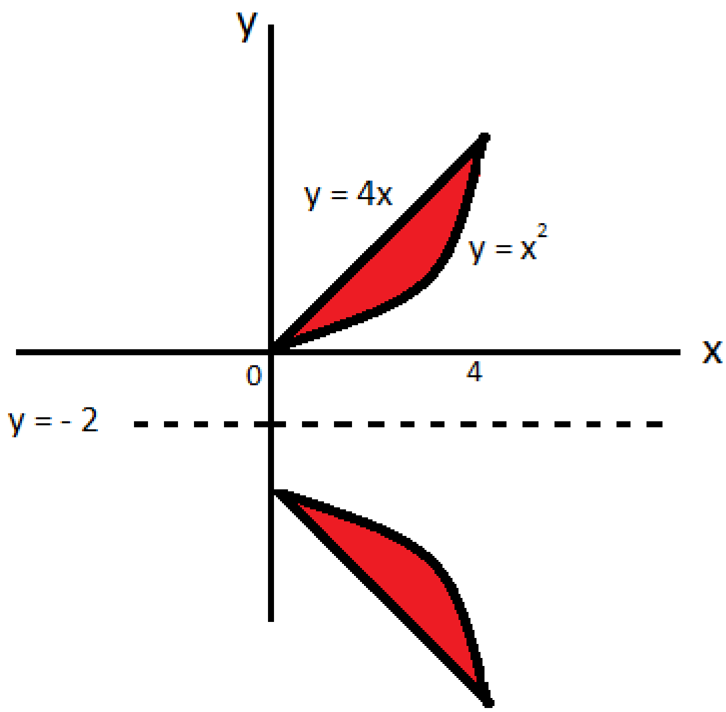
15. Determine the volume of the solid that results when the region bounded by $y = x^2$ and $y = 4x$, is revolved about the line $y = -2$.

You are not given the limits of integration, so you need to find where the two curves intersect by setting the equations equal to each other.

$y = x^2$ and $y = 4x$: $\quad x^2 = 4x \quad \rightarrow \quad x^2 - 4x = 0$

So, $x = 0, 4$. These will be our limits of integration.

Now, sketch the curve:



Notice that the distance from the axis of revolution is no longer found by just using each equation. Now you need to add 2 to each equation to account for the shift in the axis.

Thus the radii are $x^2 + 2$ and $4x + 2$.

This means that we need to evaluate the integral:

$$V = \int_{x=0}^{x=4} \pi\left(\left(4x+2\right)^2 - \left(x^2+2\right)^2\right)dx$$

$$= \pi\int_0^4 \left(12x^2 + 16x - x^4\right)dx = \pi\left[4x^3 + 8x^2 - \frac{1}{5}x^5\right]_0^4 = \pi\left(256 + 128 - \frac{1024}{5}\right)$$

$$= \pi\left(384 - \frac{1024}{5}\right) = \frac{896\pi}{5} = 179.2\pi \text{ cubic units.}$$

16. $\begin{cases}\text{Determine the volume of the solid that results when} \\ \text{the region bounded by } y = 2\sqrt{x},\ x = 4 \text{ and } y = 0, \text{ is} \\ \text{revolved around the y-axis (use cylindrical shells).}\end{cases}$

First, draw a diagram and note that the thickness will be dx.

$$V = 2\pi r \bullet \text{thickness}$$

$$V = 2\pi \int_0^4 x\left(2\sqrt{x}\right) dx$$

$$V = 4\pi \int_0^4 x^{\frac{3}{2}}\, dx = \left(\frac{8\pi}{5} x^{\frac{5}{2}}\right)_0^4 = \frac{256\pi}{5} \text{ cubic units}$$

17. $\begin{cases}\text{Determine the volume of the solid that results when} \\ \text{the region bounded by } y = x^3,\ x = 2 \text{ and the x-axis,} \\ \text{is revolved around the line } x = 2.\end{cases}$

First, draw the diagram, and note that the height of the disk = dy.

$$V = \pi r^2 h \text{ for a disk.}$$

$$V = \pi \int_{y=0}^{y=8} \left(2 - \sqrt[3]{y}\right)^2 dy$$

$$V = \pi \int_0^8 \left(4 - 4\sqrt[3]{y} + y^{\frac{2}{3}}\right) dy = \pi\left[4y - 3y^{\frac{4}{3}} + \frac{3}{5} y^{\frac{5}{3}}\right]_0^8$$

$$= \pi\left(32 - 48 + \frac{96}{5}\right) = \frac{16\pi}{5} \text{ cubic units}$$