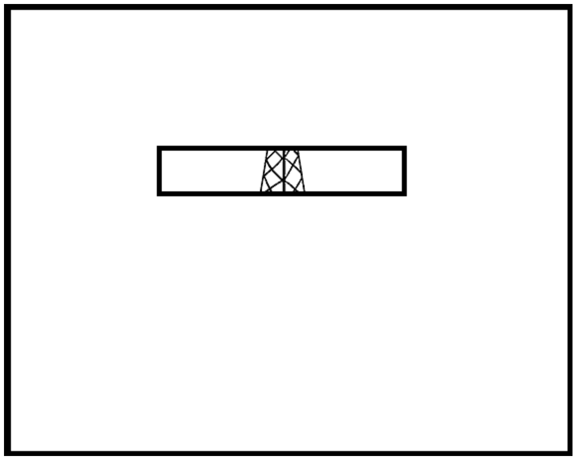
A Turvy for A.P. Calculus Exam

Puzzle and Answer Key by David Pleacher



Caption for the picture:

"E I F F E L   T O W E R   A S   S E E N   B Y
9 2 3 3 9 12   10 15 4 9 11   1 17   17 9 9 18   5 14

G U A R D   I N   A R M O R E D   T R U C K."
7 13 1 11 16   2 18   1 11 19 15 11 9 16   10 11 13 21 22

Caption for the picture turned upside down:

"E I F F E L   T O W E R   A S   S E E N   B Y
9 2 3 3 9 12   10 15 4 9 11   1 17   17 9 9 18   5 14

G U A R D   I N   A R M O R E D   T R U C K
7 13 1 11 16   2 18   1 11 19 15 11 9 16   10 11 13 21 22

W H I C H   H A S   J U S T   B E E N
4 8 2 21 8   8 1 17   20 13 17 10   5 9 9 18

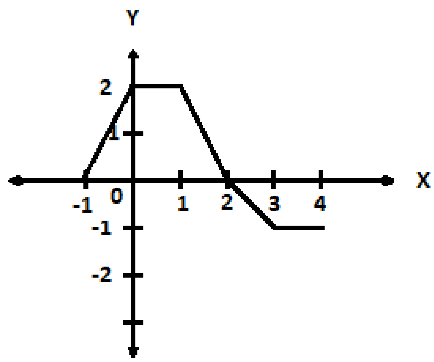
O V E R T U R N E D   B Y   A   G A N G   O F
15 6 9 11 10 13 11 18 9 16   5 14   1   7 1 18 7   15 3

T H I E V E S."
10 8 2 9 6 9 17

A 1. What is the x-coordinate of the point of inflection on the graph of $y = \frac{1}{3}x^3 + 5x^2 + 24$?

$$y = \frac{1}{3}x^3 + 5x^2 + 24$$
$$y' = x^2 + 10x$$
$$y'' = 2x + 10$$
$$\text{Set } 2x + 10 = 0$$
$$\therefore\ x = -5$$

Check concavity on either side to make sure they are different.



I 2. The graph of a piecewise-linear function *f*, for $-1 \le x \le 4$, is shown above.

What is the value of $\int_{-1}^{4} f(x)\,dx$?

The area under the curve will be the answer where the part of the graph above the x-axis is positive and the part of the graph below the x-axis will be negative:

$\int_{-1}^{4} f(x)\,dx$ = (1 + 2 + 1) + (-.5 + -1) = 2.5

F 3. $\int_{1}^{2} \frac{1}{x^2}\,dx = \int_{1}^{2} x^{-2}\,dx = -x^{-1}\Big|_1^2 = -\frac{1}{2} - (-1) = .5$

W 4. $\int_{0}^{x} \sin t\,dt = -\cos t\Big|_0^x = -\cos x + \cos 0 = 1 - \cos x$

B 5. If $x^2 + xy = 10$, then when

$x = 2,\ y = 3$

Differentiating implicitly, $2x + x\dfrac{dy}{dx} + y = 0$

$2 \bullet 2 + 2\dfrac{dy}{dx} + 3 = 0 \quad 2\dfrac{dy}{dx} = -7 \quad \dfrac{dy}{dx} = -3.5$

V 6. $\displaystyle\int_1^e \left(\frac{x^2-1}{x}\right) dx = \int_1^e \left(x - \frac{1}{x}\right) dx = \left(\frac{x^2}{2} - \ln x\right)\Bigg|_1^e =$

$\left(\dfrac{e^2}{2} - \ln e\right) - \left(\dfrac{1^2}{2} - \ln 1\right) = \dfrac{e^2}{2} - 1 - \dfrac{1}{2} = \dfrac{e^2}{2} - \dfrac{3}{2}$

G 7. Let f and g be differentiable functions with the following properties:

(i) $g(x) > 0$ for all x

(ii) $f(0) = 1$

If $h(x) = f(x)g(x)$ and $h'(x) = f(x)g'(x)$, then $f(x) =$

$h(x) = f(x)g(x)$ and $h'(x) = f(x)g'(x)$

Product Rule: $h'(x) = f(x)g'(x) + f'(x)g(x)$

$f(x)g'(x) = f(x)g'(x) + f'(x)g(x)$

$\therefore f'(x)g(x) = 0$

Since, $g(x) > 0$, then $f'(x) = 0$.

Hence, $f(x)$ must equal a constant and since $f(0) = 1$

that means $f(x) = 1$

H 8. What is the instantaneous rate of change at x = 2 of the function *f* given by

$f(x) = \dfrac{x^2 - 2}{x - 1}$?

$$f'(x) = \frac{(x-1)\bullet 2x - (x^2-2)\bullet 1}{(x-1)^2}$$

$$f'(2) = \frac{4-2}{1} = 2$$

E 9. If f is a linear function and 0 < a < b, then $\int_a^b f''(x)\,dx =$

Since f is linear, $f'(x)$ is zero. $\int_a^b f''(x)\,dx = \left(f'(x)+k\right)\Big|_a^b = \left(0+k\right)\Big|_a^b = 0$

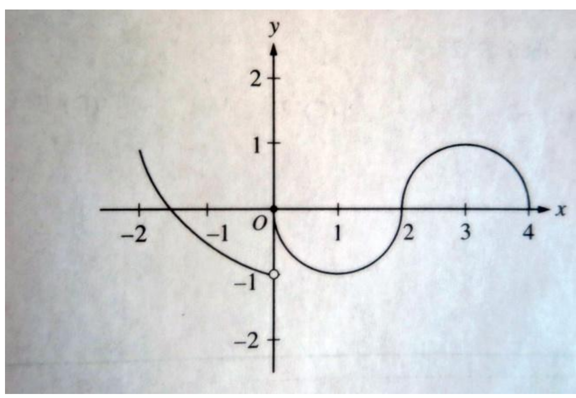
T 10. If $f(x) = \begin{cases} \ln x & \text{for } 0 < x \le 2 \\ x^2 \ln 2 & \text{for } 2 < x \le 4, \end{cases}$ then $\lim_{x\to 2} f(x)$ is

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \ln x = \ln 2$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2+} x^2 \ln 2 = 4\ln 2$

Since $\lim_{x\to 2^-} f(x) \ne \lim_{x\to 2^+} f(x)$

Therefore, the limit does not exist.



R 11. The graph of the function f shown in the figure above has a vertical tangent at the point (2, 0) and horizontal tangents at the points (1, -1) and (3, 1). For what values of x, -2 < x < 4, is f not differentiable?

If a function is differentiable, then it must be continuous (converse is not true).

So this function is NOT differentiable at x = 0.

If a function has a vertical tangent, it is not differentiable, so this function is NOT differentiable at x = 2.

L 12. A particle moves along the x-axis so that its position at time t is given by $x(t)=t^2-6t+5$. For what value of t is the velocity of the particle zero?

$v=x'(t)=2t-6$

so $2t-6=0$ and $t=3$

U 13. If $f(x)=\sin\left(e^{-x}\right)$, then

$$f'(x)=\cos\left(e^{-x}\right)\bullet\frac{d}{dx}\left(e^{-x}\right)=\cos\left(e^{-x}\right)\bullet\left(-e^{-x}\right)=-e^{-x}\cos\left(e^{-x}\right)$$

Y 14. An equation of the line tangent to the graph of $y=x+\cos x$ at the point (0, 1) is

$y'=1-\sin(x)$

At the point (0,1), the slope is $m = y'=1\text{-}0=1$

So the equation of the tangent is $y-1=x$

O 15. If $f''(x)=x(x+1)(x-2)^2$, then the graph of f has inflection points when x =

Set equal to zero and check concavity in between the points.

Set $f''(x)=x(x+1)(x-2)^2=0$

$x=0,-1,$ and $2,$ so these are possible points of inflection.

Substitute values -10, -.5, 1, and 10 for x to check concavity:

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        +                    _            +                 +
--------------------------|-----------|------------------|-------------------
                         -1           0                  2
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Only -1 and 0 are points of concavity because the concavity did not change around x=2

D 16. What are all values of k for which $\int_{-3}^{k} x^2dx=0$?

$$\int_{-3}^{k} x^2dx=\frac{1}{3}x^3\Big|_{-3}^{k}=\frac{k^3}{3}+9=0$$

$k^3=-27, \quad k=-3$

S 17. The function f is given by $f(x)=x^4+x^2-2$. On what interval is f increasing?

$f(x)=x^4+x^2-2$

$f'(x)=4x^3+2x$ Setting the first derivative equal to zero gives only x = 0.

$f''(x)=12x^2+2$

The second derivative is always positive so the function is always concave up.

Substitute points on either side of x = 0 in the first derivative to see whether the function is increasing or decreasing. For x > 0, it is increasing.

N 18. The maximum acceleration attained on the interval $0 \le t \le 3$ by the particle whose velocity is given by $v(t)=t^3-3t^2+12t+4$ is

Take the derivative of the velocity to get the acceleration:

$v(t)=t^3-3t^2+12t+4$

$a(t)=v'(t)=3t^2-6t+12$

Take the derivative of the acceleration to get the jerk and find out where possible relative maximums are:

$j(t)=6t-6$

Set $6t-6=0, \quad t=1.$

So check the values of the acceleration at $t=0, 1,$ and $3.$

$t=0, \quad a=12$

$t=1, \quad a=9$

$t=3, \quad a=21$

M 19. What is the area of the region between the graphs of $y=x^2$ and $y=-x$ from X = 0 to x = 2?

$$A=\int_0^2\left(x^2-(-x)\right)dx=\int_0^2\left(x^2+x\right)dx=\left.\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\right|_0^2=\frac{8}{3}+2-0=\frac{14}{3}$$

_J_ 20. What is the average value of $y = x^2\sqrt{x^3+1}$ on the interval [0, 2] ?

$$\bar{y} = \frac{1}{2-0}\int_0^2 \left(x^2\sqrt{x^3+1}\right)dx$$

$$\text{Let } u = x^3+1$$

$$\text{Then } du = 3x^2dx$$

$$\int\left(x^2\sqrt{x^3+1}\right)dx = \frac{1}{3}\int\left(3x^2\sqrt{x^3+1}\right)dx = \frac{1}{3}\int\left(\sqrt{u}\right)du = \frac{1}{3}\bullet\frac{2}{3}u^{\frac{3}{2}}$$

$$\therefore \bar{y} = \frac{1}{2}\bullet\frac{2}{3}\left(x^3+1\right)^{\frac{3}{2}}\Big|_0^2 = \frac{1}{9}(27-1) = \frac{26}{9}$$

_C_ 21. If $f(x) = \cot 2x$, then $f'\left(\frac{\pi}{6}\right) =$

$$f(x) = \cot 2x$$

$$f'(x) = -\csc^2(2x)\bullet\frac{d}{dx}(2x) = -2\csc^2(2x)$$

$$f'\left(\frac{\pi}{6}\right) = -2\csc^2\left(2\bullet\frac{\pi}{6}\right) = -2\csc^2\left(\frac{\pi}{3}\right) = -2\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{-8}{3}$$

_K_ 22. Let *F(x)* be an antiderivative of $\frac{(\ln x)^3}{x}$. If *F(1)* = 0, then *F(9)* =

Use a calculator to solve:

$$F(x) = \int_1^9 \frac{(\ln x)^3}{x}dx = 5.8269$$

$$\text{Since } F(1) = 0, \quad \text{Then } F(9) = 5.8269$$

Answers: (units have been omitted)

A. -5	J. $\frac{26}{9}$	S. $(0, \infty)$
B. -3.5	K. 5.827	T. Nonexistent
C. $\frac{-8}{3}$	L. 3	U. $-e^{-x}\cos\left(e^{-x}\right)$
D. -3	M. $\frac{14}{3}$	V. $\frac{e^2}{2}-\frac{3}{2}$
E. 0	N. 21	W. $1-\cos x$
F. 0.5	O. -1 and 0 only	X. $1+\cos x$
G. 1	P. 0, 1, and 3 only	Y. y = x + 1
H. 2	Q. -1, 0, and 2 only	Z. None of the above
I. 2.5	R. 0 and 2 only	