Turvy for Integration -- Solution Key

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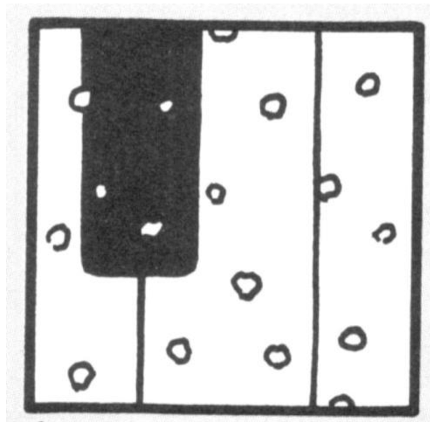

Here is the title right-side-up: Cookie crumbs on a piano

Here is the title upside-down: Abe Lincoln walking by a tall fence in a snowstorm

1. Evaluate $\int\left(\frac{1}{x^4}+\frac{1}{x^2}+x^{10}\right)dx$

$$\int\left(\frac{1}{x^4}+\frac{1}{x^2}+x^{10}\right)dx=\int\left(x^{-4}+x^{-2}+x^{10}\right)dx=-\frac{x^{-3}}{3}-x^{-1}+\frac{x^{11}}{11}+C$$

2. $\int 4\left(\frac{1}{x}+x^{\frac{2}{5}}\right)^2 dx=$

First, rewrite so you can see all the exponents and expand to get

$$\int 4\left(\frac{1}{x}+x^{\frac{2}{5}}\right)^2 dx=\int 4\left(x^{-1}+x^{\frac{2}{5}}\right)^2 dx=\int 4\left(x^{-2}+2x^{-\frac{3}{5}}+x^{\frac{4}{5}}\right)dx=$$

$$4\int x^{-2}\,dx+8\int x^{-\frac{3}{5}}\,dx+4\int x^{\frac{4}{5}}\,dx=-\frac{4}{x}+20x^{\frac{2}{5}}+\frac{20}{9}x^{\frac{9}{5}}+K$$

3. $\int\frac{x^2-9}{x+3}dx=$

$$\int\frac{x^2-9}{x+3}dx=\int\frac{(x-3)(x+3)}{x+3}dx=\int(x-3)\,dx=\frac{x^2}{2}-3x+C$$

4. $\int(\sin x-3\cot x\sin x)\,dx=$

$$\int(\sin x-3\cot x\sin x)\,dx=\int\left(\sin x-3\frac{\cos x}{\sin x}\sin x\right)dx=$$
$$\int(\sin x-3\cos x)\,dx=-\cos x-3\sin x+C$$

5. $\int\left(2x^{-\frac{3}{7}}+\frac{5}{\sin^2 x}\right)dx=$

$$\int\left(2x^{-\frac{3}{7}}+\frac{5}{\sin^2 x}\right)dx=\int\left(2x^{-\frac{3}{7}}+5\csc^2 x\right)dx=\frac{7}{2}x^{\frac{4}{7}}-5\cot x+C$$

6. $\int\csc^2 x\cos x\,dx=$

$$\int\csc^2 x\cos x\,dx=\int\frac{1}{\sin^2 x}\cos x\,dx=\int\frac{1}{\sin x}\frac{\cos x}{\sin x}\,dx=$$
$$\int\csc x\cot x\,dx=-\csc x+C$$

7. $\int\frac{d}{dx}\left(3x^{-2}+\tan x-4\right)dx=$

You are looking for the antiderivative of $\frac{d}{dx}\left(3x^{-2}+\tan x-4\right)$.

An antiderivative is a function $F(x)$, such that $\frac{d}{dx}\left(F(x)\right)=\frac{d}{dx}\left(3x^{-2}+\tan x-4\right)$.

The family of functions that satisfies this is
$3x^{-2}+\tan x-4+C$, or more simply $3x^{-2}+\tan x+K$

8. $\int\cos(2\text{x})\sqrt{\sin(2\text{x})}\,dx=$

Let $u=\sin(2x)$.
Then $du=2\cos(2x)dx$.
Now, rewriting the integral you get

$$\int\cos(2\text{x})\sqrt{\sin(2\text{x})}\,dx=\int\frac{1}{2}\left(\sin(2\text{x})\right)^{\frac{1}{2}}2\cos(2\text{x})\,dx=$$

$$\int \frac{1}{2} u^{\frac{1}{2}}\, du = \frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3} u^{\frac{3}{2}} + C = \frac{1}{3} \sin^{\frac{3}{2}}(2x) + C$$

9. $\int \frac{x}{\sqrt{1-x^2}}\, dx =$

Let $u = 1 - x^2$, then $du = -2x\, dx$

$$\int \frac{x}{\sqrt{1-x^2}}\, dx = \frac{-1}{2} \int -2x\left(1-x^2\right)^{\frac{-1}{2}} dx =$$

$$\frac{-1}{2} \int (u)^{\frac{-1}{2}} du = \frac{-1}{2} \bullet 2u^{\frac{1}{2}} + C =$$

$$-u^{\frac{1}{2}} + C = -\sqrt{1-x^2} + C$$

10. Given that $g'(x) = (\sin x)(5 + 5\cos x)^3$,
find $g(x)$ if $g(0) = 0$

Given that $g'(x) = (\sin x)(5 + 5\cos x)^3$,

Then $g(x) = \int (\sin x)(5 + 5\cos x)^3\, dx$

Let $u = 5 + 5\cos x$ then $du = -5\sin x\, dx$

$$\int (\sin x)(5 + 5\cos x)^3\, dx = \frac{-1}{5} \int (5 + 5\cos x)^3 (-5)(\sin x)\, dx =$$

$$\frac{-1}{5} \int (u)^3\, du = \frac{-1}{20} u^4 + C = -\frac{1}{20}(5 + 5\cos x)^4 + C.$$

Now use the given conditions that $g(0) = 0$ to find $g(x)$:

$$g(x) = -\frac{1}{20}(5 + 5\cos x)^4 + C, \quad \text{so}$$

$$g(0) = -\frac{1}{20}(5 + 5\cos(0))^4 + C$$

$$0 = -\frac{1}{20}(5 + 5(1))^4 + C$$

$$\therefore \quad C = 500$$

$$\therefore \quad g(x) = -\frac{1}{20}(5 + 5\cos x)^4 + 500$$

11. $\int 8x\left(x^2+7\right)^3 dx =$

Let $u = x^2+7$, then $du = 2x\,dx$

$\int 8x\left(x^2+7\right)^3 dx = 4\int \left(x^2+7\right)^3 2x\,dx =$

$4\int (u)^3 du = u^4 + C = \left(x^2+7\right)^4 + C$

12. $\int \frac{x^2}{\left(2x^3-12\right)^4} dx =$

Let $u = 2x^3 - 12$ Then $du = 6x^2 dx$

$\int \frac{x^2}{\left(2x^3-12\right)^4} dx = \int \left(2x^3-12\right)^{-4} x^2\,dx =$

$\frac{1}{6}\int \left(2x^3-12\right)^{-4} 6x^2\,dx = \frac{1}{6}\int (u)^{-4} du =$

$\frac{1}{6}\left(\frac{u^{-3}}{-3}\right) + C = \frac{-1}{18(u)^3} + C = \frac{-1}{18\left(2x^3-12\right)^3} + C$

13. $\int \frac{\left(2+\sqrt{x}\right)^6}{\sqrt{x}} dx =$

Let $u = 2 + x^{\frac{1}{2}}$ Then $du = \frac{1}{2}(x)^{-\frac{1}{2}} dx$

$\int \frac{\left(2+\sqrt{x}\right)^6}{\sqrt{x}} dx = \int \left(2+x^{\frac{1}{2}}\right)^6 (x)^{-\frac{1}{2}} dx =$

$\int 2\left(2+x^{\frac{1}{2}}\right)^6 \frac{1}{2}(x)^{-\frac{1}{2}} dx = \int 2(u)^6 du =$

$\frac{2}{7}(u)^7 + C = \frac{2}{7}\left(2+x^{\frac{1}{2}}\right)^7 + C$

14. $\int\left(3-\frac{1}{x}\right)^{-2}\left(\frac{1}{x^2}\right)dx=$

Let $u=3-\frac{1}{x}$ Then $du=\frac{1}{x^2}dx$

$$\int\left(3-\frac{1}{x}\right)^{-2}\left(\frac{1}{x^2}\right)dx=\int u^{-2}du=$$

$$-(u)^{-1}+K=-\left(3-\frac{1}{x}\right)^{-1}+K$$

15. $\int\frac{\cos x}{(2-3\sin x)^4}dx=$

Let $u=2-3\sin x$ Then $du=-3\cos x\,dx$

$$\int\frac{\cos x}{(2-3\sin x)^4}dx=\int-\frac{1}{3}(2-3\sin x)^{-4}(-3\cos x\,dx)=$$

$$\int-\frac{1}{3}(u)^{-4}du=\frac{1}{9}(u)^{-3}+C=\frac{1}{9}(2-3\sin x)^{-3}+C$$

16. $\int\frac{x}{\cos^2(3x^2)}dx=$

Use the trig identity, $\frac{1}{\cos x}=\sec x$:

$$\int\frac{x}{\cos^2(3x^2)}dx=\int\sec^2(3x^2)\,x\,dx$$

Let $u=3x^2$ Then $du=6x\,dx$

So, $\int\sec^2(3x^2)\,x\,dx=\int\frac{1}{6}\sec^2(3x^2)\,6x\,dx=$

$$\int\frac{1}{6}\sec^2(u)\,du=\frac{1}{6}\tan(u)+C=\frac{1}{6}\tan(3x^2)+C$$

17. $\int \frac{4x^3 - 3}{\left(x^4 - 3x\right)^2}\,dx =$

Let $u = x^4 - 3x$ Then $du = \left(4x^3 - 3\right)dx$

$$\int \frac{4x^3 - 3}{\left(x^4 - 3x\right)^2}\,dx = \int \frac{du}{(u)^2} = \int u^{-2}du =$$

$$-1u^{-1} + C = \frac{-1}{(u)} + C = \frac{-1}{\left(x^4 - 3x\right)} + C$$

18. $\int \cos x \cos(\sin x)\; dx =$

Let $u = \sin x$ Then $du = \cos x\, dx$

$$\int \cos x \cos(\sin x)\; dx = \int \cos(\sin x)\cos x\, dx =$$

$$\int \cos(u)\,du = \sin u + C = \sin(\sin x) + C$$

19. $\int 3x \tan\left(3x^2\right)\sec^2\left(3x^2\right)\; dx =$

This integral can be done two different ways:

(1) Let $u = \tan\left(3x^2\right)$, Then $du = \sec^2\left(3x^2\right)6x\,dx$

$$\int 3x \tan\left(3x^2\right)\sec^2\left(3x^2\right)\; dx = \int \frac{1}{2}\tan\left(3x^2\right)\sec^2\left(3x^2\right)\; 6x\,dx =$$

$$\int \frac{1}{2}(u)\,du = \frac{1}{4}u^2 + C = \frac{1}{4}\tan^2\left(3x^2\right) + C$$

(2) Let $u = \sec\left(3x^2\right)$, Then $du = \sec\left(3x^2\right)\tan\left(3x^2\right)\; 6x\,dx$

$$\int 3x \tan\left(3x^2\right)\sec^2\left(3x^2\right)\; dx = \int \frac{1}{2}\sec\left(3x^2\right)\sec\left(3x^2\right)\tan\left(3x^2\right)\; 6x\,dx =$$

$$\int \frac{1}{2}(u)\,du = \frac{1}{4}u^2 + K = \frac{1}{4}\sec^2\left(3x^2\right) + K$$

These answers look different but they are actually equivalent because of the identity $1 + \tan^2 x = \sec^2 x$, so the constants K and C differ by 1.