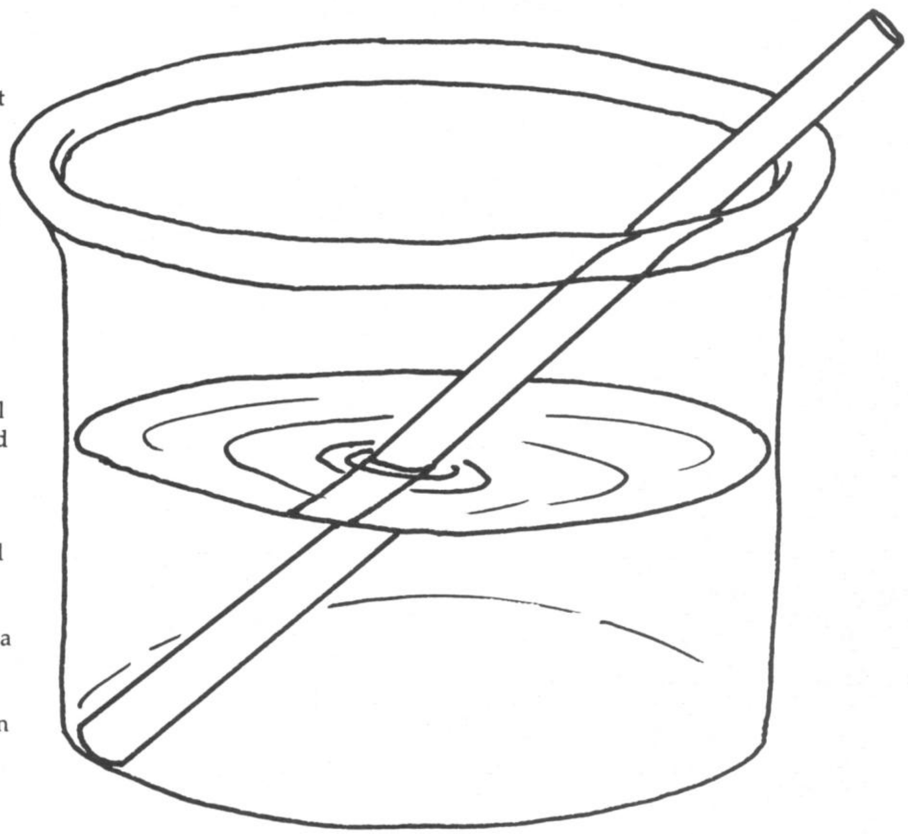
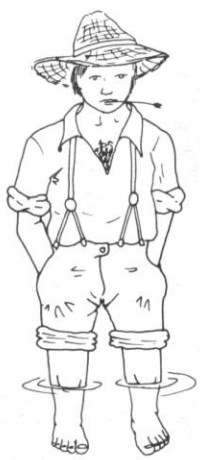
Bending Light

Richard G. Montgomery

Light bends! Light travels at different speeds through different substances and changes direction as it changes speed. A pole lowered into a pool of water appears to bend. A man standing in waist-deep water appears shorter.



In this Pull-Out Lesson, we will discuss the mathematics behind the phenomenon of bending light. We will look at Snell's Law which relates the size of the bend to the change in speed as light passes from one substance to another. The student will be able to see why a window does not appear to bend the scenery even though light travels slower in glass than in air. Does sound bend also?



Bending light is not twisting the facts. A mirror provides evidence that light bounces, but light also bends. Evidence of this is the familiar "bent-pole" effect seen when a pole is placed into still water, or your friend stands in a calm swimming pool. His legs look shortened because they, or rather their image, is bending as it passes from water into air. This is an example of the refraction of light and results from differing speeds of light.

Your-Turn! **(Number 1)** Get a pole and find a half-filled pool of water. (I used the handle of a wooden mixing spoon and the bathroom sink.) Hold the pole vertically and slowly lower it into the water. See it shrink. Lean the pole to one side. See it bend.

Light travels at different speeds in different substances. It travels slower through water than through air; slower through air than through almost empty space. The c in Einstein's famous equation $E = mc^2$ is the assumed constant speed of light in perfectly empty space (about 186,000 miles per second). Light changes direction as it changes speed while passing from one medium into another. Snell's Law is an equation which neatly relates the size of the bend to the change in speed.

To explain Snell's Law, suppose there are two adjacent mediums, A and B, separated by a perfectly straight boundary of no width (stretch your imagination). Also suppose that light travels at a constant velocity v_1 in medium A and constant velocity v_2 in medium B. (See Figure 1.) Next imagine a ray of light approaching the boundary at angle α (measured from the perpendicular). (See Figure 1a.) As it crosses the boundary it bends and departs at angle β (again measured from the perpendicular). (See Figure 1b.) Snell's Law relates the values of α, β, v_1, v_2 as follows.

Snell's Law: $(\sin \alpha)/(\sin \beta) = v_1/v_2$.

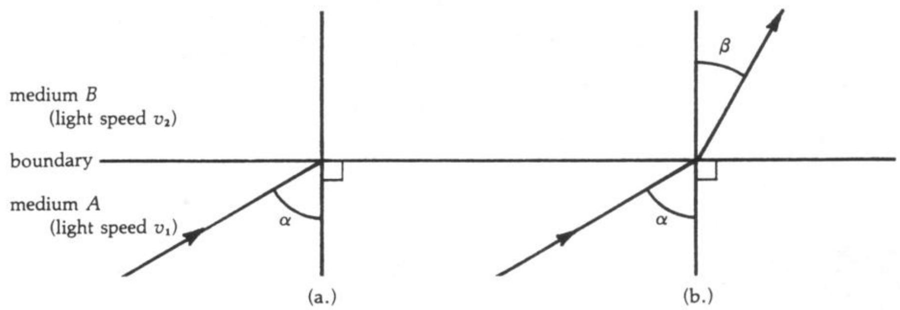


Figure 1.

To see this law in action let's suppose that light travels 1.5 times faster in medium B than in medium A: thus $v_2 = 1.5v_1$. Next let's compute the value of departing angle β for several different values of approach angle α. First rewrite Snell's equation as

$$\sin \beta = (v_2/v_1) \sin \alpha = 1.5 \sin \alpha$$

or

$$\beta = \sin^{-1} (1.5 \sin \alpha).$$

For $\alpha = 20°$, we have $\beta = \sin^{-1} (1.5 \sin 20°)$. On my hand calculator I enter *20*, push some buttons:

20 SIN × 1.5 = INV SIN

and read 30.865883, which is rounded to 31°. So, in this situation with $v_2 = 1.5v_1$, if a ray of light approaches the boundary at an angle of 20° (measured from the perpendicular), then it will depart at an angle of 31° (measured from the perpendicular). This information I have recorded on Figure 2.

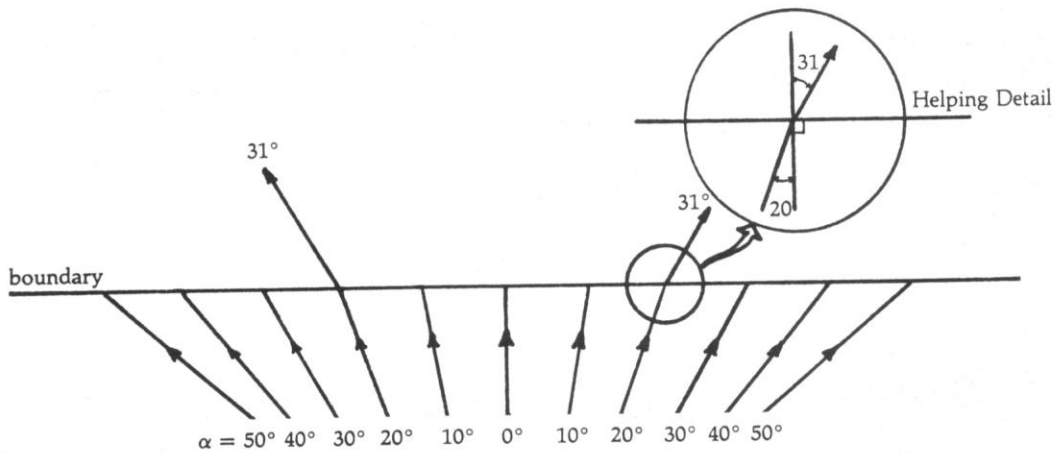


Figure 2.

Your-Turn! (**Number 2**) You do the same for each of $\alpha = 0°$, $10°$, $30°$, $40°$, and $50°$. Record your answers directly onto Figure 2. Draw and label the departing rays. Be neat, use a protractor. (Don't forget to measure your angles perpendicular to the boundary.)

What happened when you used $\alpha = 50°$? Any decent calculator should have said ERROR. Why? Because 1.5 sin50° is greater than 1, and so cannot be sin β for any β. Does this mean that a ray of light cannot approach the boundary at this angle? Of course not; instead there is a critical approach angle α_0 beyond which no light crosses the boundary. To find α_0, set $\beta = 90$ in Snell's Law and solve for α.

Your-Turn! (**Number 3**) Find the critical angle α_0 in the situation above where $v_2 = 1.5v_1$.

Figure 2 indicates that when $v_2 = 1.5v_1$, a light ray bends toward the boundary as it crosses it; that is, β is greater than α. This can be seen directly from Snell's Law.

$$(\sin \alpha)/(\sin \beta) = v_1/v_2 = v_1/(1.5v_1) = 1/1.5.$$

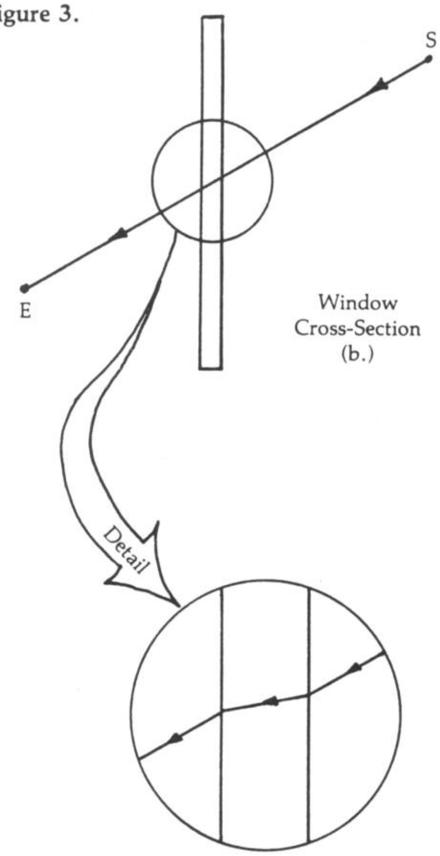
$$\text{Thus, } \sin \beta = 1.5 \sin \alpha,$$

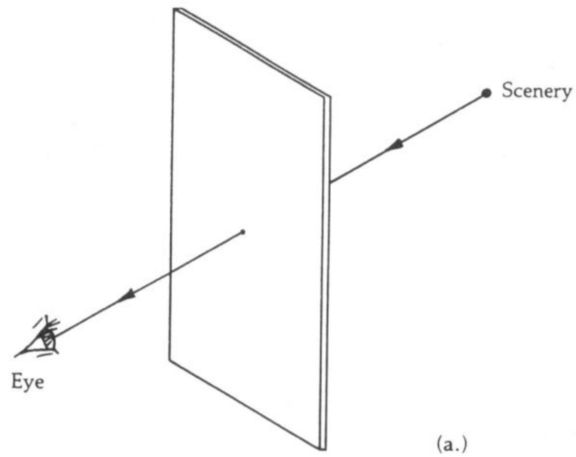
$$\text{so } \sin \beta > \sin \alpha,$$

$$\text{and so } \beta > \alpha.$$

(Note these steps use the fact that all angles here are between 0 and 90 degrees.) The argument clearly will hold not only for the case where $v_2 = 1.5v_1$, but also anytime $v_2 = kv_1$ and k *is a constant greater than* 1. These correspond to the situations where light travels slower in medium A than in medium B.

Figure 3.





Bending-Fact #1. If light travels slower in medium *A* than in medium *B*, then as a ray crosses from medium *A* into medium *B* it bends toward the boundary.

Your-Turn! (**Number 4**) Figure out what happens when light travels faster in medium *A* than in medium *B*. Does it bend toward the boundary or away from the boundary? Now that you have guessed the answer, use Snell's Law to prove yourself correct. (Proceed as we did above. First use the case with $v_2 = 0.5v_1$. Then consider the general situation with $v_2 = kv_1$, where k is a constant *between* 0 *and* 1.)

Bending-Fact #2. If light travels faster in medium *A* than in medium *B*, then as a ray crosses from medium *A* into medium *B* it bends away from the boundary.

With these Bending-Facts in hand let's use them (without twisting) to explain why a window does not appear to bend the scenery. Why should it? Because light travels slower in glass than in air. Why doesn't it? Because the window bends the scenery twice: first away from a boundary (by Bending-Fact #2), then toward a boundary (by Bending-Fact #1). (See Figure 3.) Snell's Law can be used to show that the final outgoing light rays are parallel to the incoming light rays, so the scenery looks just about like it should. In Figure 4 we want to verify that entering ray r_1 is parallel to departing ray r_2. Snell's Law gives us two equations, one for each boundary:

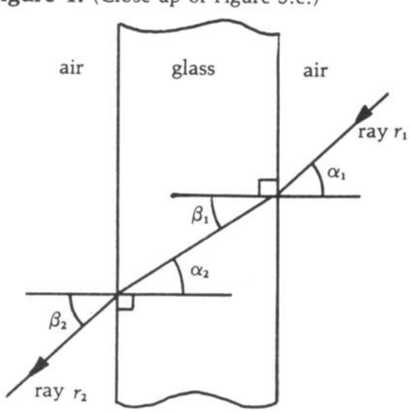
$$(\sin \alpha_1)/(\sin \beta_1) = (v_A/v_G)$$
$$\text{and } (\sin \alpha_2)/(\sin \beta_2) = (v_G/v_A),$$

where v_A and v_G are the speed of light in air and in glass, respectively.

$$\text{So } (\sin \alpha_1)/(\sin \beta_1) = (\sin \beta_2)/(\sin \alpha_2).$$
$$\text{But } \beta_1 = \alpha_2, \text{ so } \sin \alpha_1 = \sin \beta_2,$$
$$\text{and } \alpha_1 = \beta_2,$$

which means that ray r_1 is parallel to ray r_2 as hoped.

Figure 4. (Close-up of Figure 3.c.)



Back now to your friend in the swimming pool. To explain his shrunken legs let us first be content with his big toe. (See Figure 5.) Bending-Fact #1 explains why his toe appears to be higher than it really is. Imagine a light ray traveling from the toe to your eye. As it passes from the water into the air it bends downward. But your eye doesn't know that and so it sees the toe above its actual location. Now look at your whole friend. Although he appears oddly shortened he's really all there (Whew!). It's just that all his underwater parts appear higher than they really are, and so you have one distorted friend. That's not twisting the facts, that's Bending-Facts in action.

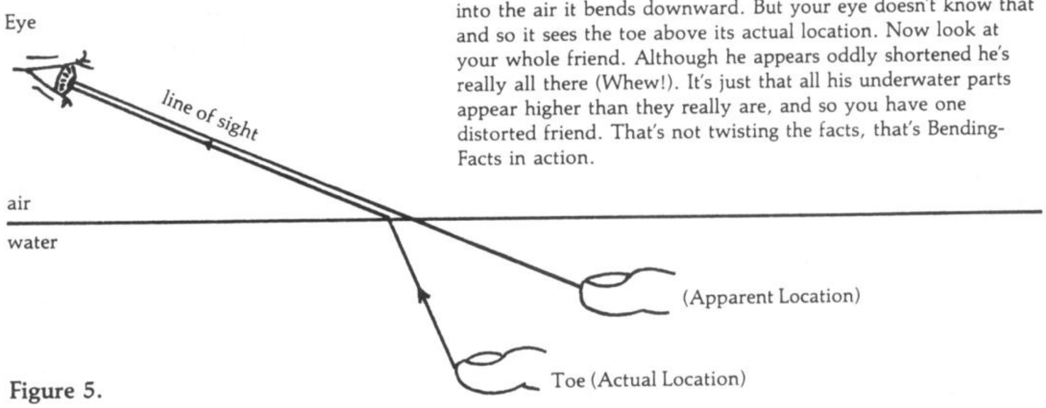


Figure 5.

Your-Turn! (Number 5) Explain how the shrink and the bend you observed with your pool and pole (in Your-Turn Number 1) are each examples of Bending Fact #1 in action.

Final Twists & Bends 'Snell' refers to the Dutch mathematician Willebrord Snell who in 1621 discovered the equation for the bend when light passes from air into water. The bend had been observed and measured much earlier (accurate tables exist from 140 A.D.). About 1650 the law was finally given a logical basis by the French mathematician Pierre de Fermat. Chapter 26 of *The Feynman Lectures* has a nice discussion and non-calculus derivation of Snell's Law as Fermat could have done it. Also found there are applications to optics and unexpected bending facts (like: when you see the sunrise, the sun is still below the horizon!).

Snell's Law is valid for sound as well as light. (Sound also bends!) *Listening to the Earth* tells how bending sound and Snell's Law are used to learn about the earth's underground. Included is a calculus derivation of the law for students who know about derivatives.

References

Richard Feynman, Robert Leighton, and Matthew Sands, *The Feynman Lectures on Physics* Volume 1, Addison-Wesley Publishing Company, 1963.

Richard Montgomery, *Listening to the Earth*, UMAP Module 292, Consortium for Mathematics and Its Applications (COMAP), 1980.

Answers

Your-Turn! Number 2.

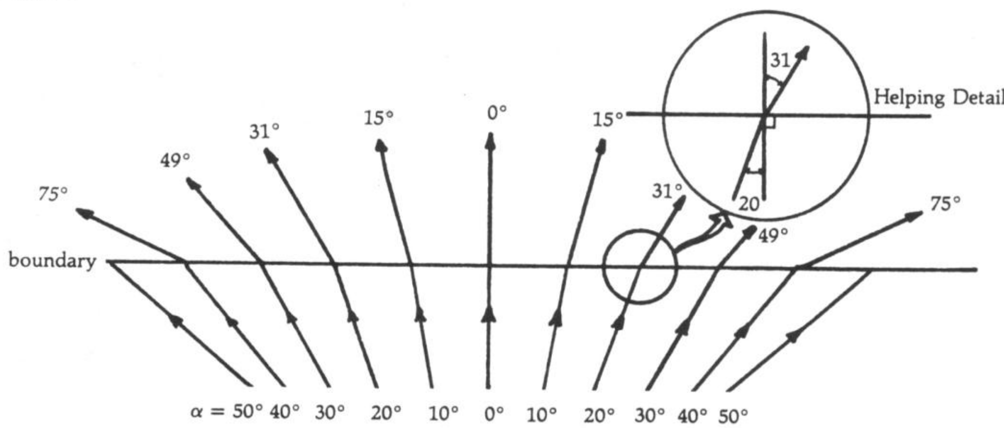


Figure 2.

Your-Turn! Number 3.

$\frac{\sin \alpha}{\sin \beta} = \frac{v_1}{v_2}$ with $\beta = 90°$

and $v_1/v_2 = 1/1.5 = 2/3$,
so $\sin \alpha = 2/3$
and $\alpha_0 = 41.8^+$ degrees.

Your-Turn! Number 4.

Let $v_2 = 0.5v_1$.
Then $\frac{\sin \alpha}{\sin \beta} = \frac{v_1}{v_2} = \frac{1}{0.5}$,
so $\sin \beta = 0.5 \sin \alpha$.
But then $\sin \beta < \sin \alpha$
and $\beta < \alpha$ as desired.
More generally,
let $v_2 = kv_1$ with $0 < k < 1$.
Then $\sin \beta = k \sin \alpha$,
so $\sin \beta < \sin \alpha$ (as $0 < k < 1$)
and $\beta < \alpha$ as desired.
Note: all values are positive as α and β are between 0 and 90.