

Problem 154

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The cross section of a gazebo is a 10 foot-by-10-foot square. The gazebo is located in an open level field.

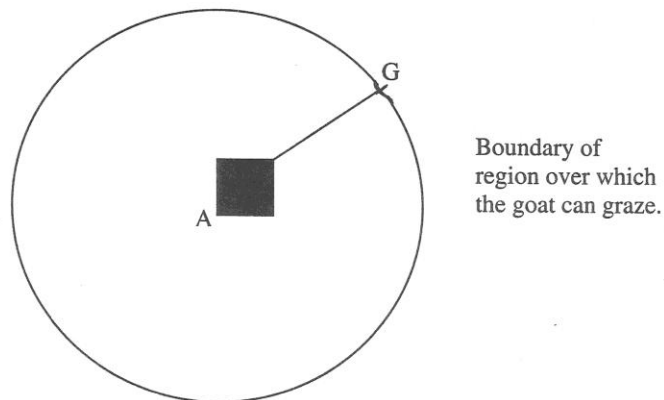
A goat is tethered to one corner of the gazebo by a forty foot rope. The goat cannot enter the gazebo.

What is the area of the region over which the goat can graze?

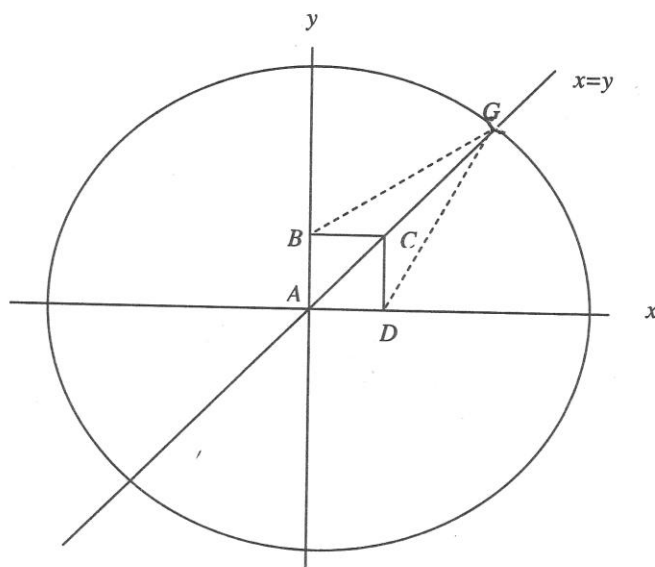
Solution by Proposer

If the goat pulls the rope taut and then walks around the gazebo, his tether will wrap around a 10ft-by-10ft square and his nose will follow the outer boundary of the region over which he can graze. The problem is to deal with the region of overlap as the goat grazes first in one sense (say, counter-clockwise) and then in the other (clockwise).

The geometry is sketched below. The goat is tethered at A . The shaded square represents the gazebo. It is at cusp G on the boundary that the grazing overlap begins as the goat winds his tether around the square. It should be clear from symmetry that segment \overline{AG} passes through the vertex of the square opposite to A .



Let us redraw the figure in the Cartesian plane with A at $(0,0)$, G in the first quadrant, and \overline{AG} on the line $x = y$. We denote the vertices of the square by A, B, C , and D .



The boundary arc in the first quadrant from the y-axis to G falls on the circle $x^2 + (y-10)^2 = 900$

We can find the coordinates of G by solving the equation of the circle and the equation of the line $x=y$ simultaneously. We find that the coordinates of G are

$$\left(\frac{10+10\sqrt{17}}{2}, \frac{10+10\sqrt{17}}{2}\right)$$

The angle between the positive y-axis and \overline{BG} is

$$\text{Arctan } \frac{\sqrt{17}+1}{\sqrt{17}-1} \text{ radians}$$

The region over which the goat can graze is composed of $\frac{3}{4}$ of a circle radius 40, two

sections with central angle $\text{Arctan } \frac{\sqrt{17}+1}{\sqrt{17}-1}$ and radius 30, and two triangles

$(\triangle ABC, \triangle ADC)$ of base 10 and altitude $\frac{10+10\sqrt{17}}{2} - 10 = 5\sqrt{17} - 5$

Thus the area of the region over which the goat can graze is

$$\frac{3}{4} \pi (40^2) + 2 \left(\frac{1}{2}\right) (30^2) \text{Arctan } \frac{\sqrt{17}+1}{\sqrt{17}-1} + 2 \left(\frac{1}{2}\right) (10)(5\sqrt{17} - 5) \doteq 4847 \text{ sq. feet}$$